GLUING OF PLANE REAL ALGEBRAIC CURVES AND CONSTRUCTIONS OF CURVES OF DEGREES 6 AND 7

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The main question of the topology of real algebraic curves is how the components of a nonsingular plane projective real algebraic curve of degree m can be positioned with respect to one another This question became well-known due to its inclusion by D.Hilbert in his famous sixteenth problem [5] . A complete answer was then known only for $m \leqslant 5$. In the late sixties D.A.Gudkov [3] completed the investigation of the case $m=6$. The answer for $m=7$ was announced in my article [8].

To resolve this question for some m it is necessary to work in the two directions: first, it is necessary to find topological restrictions imposed on a curve by its algebraic nature; second, it is necessary to find methods of construction of curves of a given degree with a prescribed topology. The works in the first direction, especially during the last decade, involve the powerful machinery of the modern topology, while the method of construction of curves remained unchanged since the XIX-th century. It consists in small perturbing a singular curve having only nondegenerate singularities (as a rule, the singular curve was a union of two nonsingular curves transversal to one another). In the case $m=7$ this method turned out to be unsufficient.

In this work a new method of construction of curves with a prescribed topology is introduced and by this method the constructions of curves of degree 7 announced in [8] are made. The new method is based on a construction that builds a new algebraic curve from several ones. From the topological point of view the new curve is arranged as a result of gluing of the initial curves. The construction can be interpreted as a perturbation of a curve with complicated singularities. Special class of such perturbations was described in my article [8] .

Besides the constructions of the curves of degree 7, in this article I describe a new simple construction of curves of degree 6. It is proved that a nonsingular curve of degree 6 with any possible mutual position of its ovals can be obtained by a small perturbation of the union of three ellipses tangent one another in two points.

1. GLUING

1.1. CHARTS OF POLYNOMIAL. The notion of chart of a polynomial plays an important role in the statements below concerning the gluing. The most natural way the introduce it involves toral surfaces. Another more elementary definition is related with the well-known description of the behaviour of an algebraic curve near the coordinate axes and at infinity. Consider at first the latter definition and then the former.

Let ℓ be a map $(\mathbb{R}\setminus 0)^2 \to \mathbb{R}^2 : (x,y) \mapsto (\ln|x|, \ln|y|)$. The restriction of ℓ to each quadrant is a diffeomorphism.

For a set $\mathcal{U}\subset\mathbb{R}^2$ and a real polynomial a of two variables let us denote the curve $\{(x,y)\in\mathcal{U} \mid a(x,y)=0\}$ by $V_{\mathcal{U}}(a)$. Recall that the Newton polygon $\Delta(a)$ of a polynomial $a(x,y)=\sum_{\omega\in\mathbb{Z}^2} a_\omega x^{\omega_1} y^{\omega_2}$ is the convex hull of the set $\{\omega\in\mathbb{Z}^2 \mid a_\omega\neq 0\}$. For a set $\Gamma\subset\mathbb{R}^2$ and a polynomial $a(x,y)=\sum_{\omega\in\mathbb{Z}^2} a_\omega x^{\omega_1} y^{\omega_2}$ the polynomial $\sum_{\omega\in\Gamma\cap\mathbb{Z}^2} a_\omega x^{\omega_1} y^{\omega_2}$ is called Γ-*truncation* of a and is denoted by a^Γ.

A curve $V_{(\mathbb{R}\setminus 0)^2}(\alpha x^\sigma + \beta y^\tau)$ with $\alpha,\beta\in\mathbb{R}$ and relatively prime σ,τ is called a *quasistraight line*. ℓ maps it onto a straight line orthogonal to $\Delta(\alpha x^\sigma+\beta y^\tau)$. Any straight line with a rational direction is an image under ℓ of some quasistraight line

A polynomial a of two variables is called *quasi-homogeneous*, if $\operatorname{Int}\Delta(a)=\emptyset$. If a is a real quasi-homogeneous polynomial of two variables, then $V_{(\mathbb{R}\setminus 0)^2}(a)$ decomposes into a union of quasi-straight lines maped by ℓ into straight lines orthogonal to $\Delta(a)$.

A real polynomial a of two variables is said to be *peripherally non-degenerate*, if for any side Γ of $\Delta(a)$ the truncation a^Γ has no factor of the form $(\alpha x^\sigma+\beta y^\tau)^k$ with $k>1$, $\alpha,\beta\in\mathbb{R}$ and g.c.d. $(\sigma,\tau)=1$. That is equivalent o to absence of a multiple component in $V_{(\mathbb{R}\setminus 0)^2}(a^\Gamma)$.

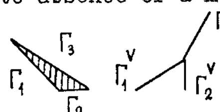


Fig.1.

For a side Γ of a convex polygon Δ let us denote by $\Gamma^\vee$ the ray orthogonal to Γ and directed outside Δ with respect to Γ. See fig.1.

The following theorem is essentially well-known and can be traced back to Newton [7].

1.1.A. Let a be a peripherally non-degenerate real polynomial of two variables and let $\Gamma_1,\dots,\Gamma_n$ be the sides of its Newton polygon $\Delta=\Delta(a)$. Then for any quadrant $Q\subset(\mathbb{R}\setminus 0)^2$ each straight line contained in $\ell(V_Q(a^{\Gamma_i}))$, $i=1,\dots,n$ is an asymptote of $\ell(V_Q(a))$ and $\ell(V_Q(a))$ goes to infinity only along these asymptotes in the directions of $\Gamma_i^\vee$.

The quasi-straight line contained in $V_Q(a^{\Gamma_i})$ is called a *logarithmic asymptote* of the curve $V_{(\mathbb{R}\setminus 0)^2}(a)$.

1.1.B. EXAMPLE. Let $a(x,y)=8x^3-x^2+4y^2$. The Newton polygon $\Delta(a)$ is shown in fig.1. The curve $V_{\mathbb{R}^2}(a)$ is shown is fig 2. The images of $V_Q(a)$ and $V_Q(a^{\Gamma_i})$ under $\ell|_Q : Q\to\mathbb{R}^2$, where Q are the quadrants, are shown in fig.3.

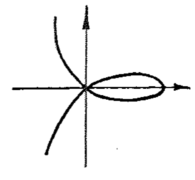


Fig.2.

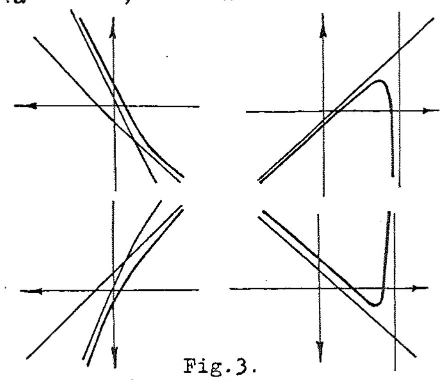


Fig.3.

For $\varepsilon,\delta=\pm 1$ let us denote by $S_{\varepsilon,\delta}$ the symmetry $\mathbb{R}^2\to\mathbb{R}^2 : (x,y)\mapsto(\varepsilon x,\delta y)$. For $A\subset\mathbb{R}^2$ let us denote $S_{\varepsilon,\delta}(A)$ by $A_{\varepsilon,\delta}$ and $A_{++}\cup A_{+-}\cup A_{-+}\cup A_{--}$ by A_*. Let us denote $\{(x,y)\in\mathbb{R}^2 \mid \varepsilon x>0,\ \delta y>0\}$ by $Q_{\varepsilon,\delta}$.

Now let us define the charts of peripherally non-degenerate polynomials. First, consider the case of a quasi-homogeneous polynomial. Let a be such polynomial and (w_1,w_2) be a vector orthogonal to $\Delta=\Delta(a)$ with integer relatively prime coordinates w_1,w_2. A pair (Δ_*,ν) consisting of Δ_* and a finite set ν will be called a *chart* of a if the number of points of $\nu\cap\Delta_{\varepsilon,\delta}$ is equal to the number of components of $V_{Q_{\varepsilon,\delta}}(a)$ for any ε,δ and if the set ν is invariant under $S_{(-1)^{w_1},(-1)^{w_2}}$ (the curve $V_{\mathbb{R}^2}(a)$ is invariant under the same symmetry).

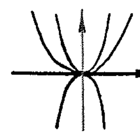


Fig.4.

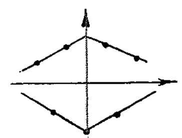


Fig.5.

1.1.C. EXAMPLE. In fig.4 a curve $V_{\mathbb{R}^2}((x^2-y)(x^2+y)(2x^2-y)y)$ is shown, and in fig 5 it is shown a chart of the polynomial $(x^2-y)(x^2+y)(2x^2-y)y$.

Now consider the case of a peripherally non-degenerate polynomial a with $\operatorname{Int}\Delta(a)\neq\emptyset$. Let $\Delta,\Gamma_1,\dots,\Gamma_n$ be as in 1.1.A. Then, as it follows from 1.1.A, there exist a disk $\mathcal{D}\subset\mathbb{R}^2$ with the centre in the origin and neighbourhoods $\mathcal{D}_1, \dots, \mathcal{D}_n$ of the rays $\Gamma_1^{\nu}, \dots, \Gamma_n^{\nu}$ such that $V_{(\mathbb{R}\setminus 0)^2}(a) \subset \ell^{-1}(\mathcal{D} \cup \mathcal{D}_1 \cup \dots \cup \mathcal{D}_n)$ and for $i = 1, \dots, n$ the curve $V_{\ell^{-1}(\mathcal{D}_i \setminus \mathcal{D})}(a)$ is contractible (in itself) into $V_{\ell^{-1}(\mathcal{D}_i \cap \partial\mathcal{D})}(a)$. A pair (Δ_*, ν) consisting of Δ_* and a curve $\nu \subset \Delta_*$ will be called a c h a r t of a_{Γ}, if: (i) for $i = 1, \dots, n$ the pair $(\Gamma_{i*}, \Gamma_{i*} \cap \cap\, \nu)$ is a chart of a^{Γ_i} and (ii) for $\varepsilon, \delta = \pm 1$ there exists a homeomorphism $h_{\varepsilon,\delta} : \mathcal{D} \to \Delta$ such that $\nu \cap \Delta_{\varepsilon,\delta} = S_{\varepsilon,\delta} \circ h_{\varepsilon,\delta} \circ \ell(V_{\ell^{-1}(\mathcal{D}) \cap Q_{\varepsilon,\delta}}(a))$ and $h_{\varepsilon,\delta}(\partial\mathcal{D} \cap \mathcal{D}_i) \subset \Gamma_i$ for $i = 1, \dots, n$.

1.1.D. EXAMPLE. A chart of $8x^3 - x^2 + 4y^2$ (see 1.1.B) is shown fig.6.

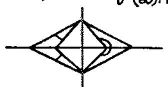


Fig.6

Roughly speaking, a chart of a is obtained from the pair $((\mathbb{R}\setminus 0)^2, V_{(\mathbb{R}\setminus 0)^2}(a))$ by removing a peripheral partof $(\mathbb{R}\setminus 0)^2$ in which $V_{(\mathbb{R}\setminus 0)^2}(a)$ is approximated by its logarithmic asymptotes and by enclosing the rest into Δ_*.

Now consider another definition of the charts. To any convex closed polygon Δ with vertices having integer coordinates it is associated a real algebraic surface $\mathbb{R}\Delta$ (see [2, 5.8]), which is a completion of $(\mathbb{R}\setminus 0)^2$. The complement $\mathbb{R}\Delta \setminus (\mathbb{R}\setminus 0)^2$ consists of straight lines corresponding to the sides of Δ. From the topological point of view $\mathbb{R}\Delta$ can be obtained from 4 copies of Δ by gluing their sides in pairs. For a real polynomial a of two variables let us denote the closure of $V_{(\mathbb{R}\setminus 0)^2}(a)$ in $\mathbb{R}\Delta$ by $V_{\mathbb{R}\Delta}(a)$.

Let a be a peripherally non-degenerate real polynomial with $\operatorname{Int} \Delta(a) \neq \emptyset$. Then $V_{\mathbb{R}\Delta(a)}(a) \setminus (\mathbb{R}\setminus 0)^2$ consists of non-singular points of $V_{\mathbb{R}\Delta(a)}(a)$ and $V_{\mathbb{R}\Delta(a)}(a)$ is transversal to the lines constituting $\mathbb{R}\Delta(a) \setminus (\mathbb{R}\setminus 0)^2$. (See, e.g., [6]). Cut $\mathbb{R}\Delta(a)$ along these lines. We obtain 4 polygons. Their interiors are naturally the quadrants $Q_{\varepsilon,\delta} \subset (\mathbb{R}\setminus 0)^2$. The polygons themselves are homeomorphic to $\Delta(a)$. Identifying them with $\Delta(a)_{\varepsilon,\delta}$ we obtain a chart of a.

It can be shown that for any peripherally non-degenerate real polynomial a and any convex polygon Δ the pair $(\mathbb{R}\Delta, V_{\mathbb{R}\Delta}(a))$ can be restored by a chart of a.

1.2. GLUING OF CHARTS. Let $a_1, \dots, a_s$ be peripherally non-degenerate real polynomials of two variables with $\operatorname{Int} \Delta(a_i) \cap \operatorname{Int} \Delta(a_j) = \emptyset$ for $i \neq j$. A pair $(\Delta_*, \nu)_s$ is said to be obtained by gluing of charts of $a_1, \dots, a_s$ if $\Delta = \bigcup_{i=1}^{s} \Delta(a_i)$ and there exist charts $(\Delta(a_i)_*, \nu_i)$ of $a_1, \dots, a_s$ such that $\nu = \bigcup_{i=1}^{s} \nu_i$.

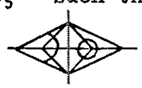


Fig.7 Fig.8

1.2.A. EXAMPLE. In fig.6 anf fig 7. charts of $8x^3 - x^2 + 4y^2$ and $4y^2 - x^2 + 1$ are shown. In fig.8 the result of gluing

of these two charts is shown.

1.3. GLUING OF POLYNOMIALS. Let $a_1, \dots, a_s$ be real polynomials of two variables with $a_i^{\Delta(a_i) \cap \Delta(a_j)} = a_j^{\Delta(a_i) \cap \Delta(a_j)}$ and $\operatorname{Int} \Delta(a_i) \cap \operatorname{Int} \Delta(a_j) = \emptyset$ for $i \neq j$. Suppose the set $\Delta = \bigcup_{i=1}^{s} \Delta(a_i)$ is convex. Let $\nu : \Delta \to \mathbb{R}$ be a non-negative convex function such that: (1) $\nu|_{\Delta(a_i)}$ is linear for $i = 1, \dots, s$; (2) if the restriction of ν to some set is linear then the set is contained in some $\Delta(a_i)$; (3) $\nu(\Delta \cap \mathbb{Z}^2) \subset \mathbb{Z}$.

There obviously exists a unique polynomial a with $\Delta(a) = \Delta$ and $a^{\Delta(a_i)} = a_i$ for $i = 1, \dots, s$. If $a(x,y) = \sum_{\omega \in \mathbb{Z}^2} a_\omega x^{\omega_1} y^{\omega_2}$, then we set $b_t(x,y) = \sum_{\omega \in \mathbb{Z}^2} a_\omega x^{\omega_1} y^{\omega_2} t^{\nu(\omega_1, \omega_2)}$ and say that the polynomials b_t are obtained by gluing of $a_1, \dots, a_s$ by ν.

1.3.A. EXAMPLE. Let $a_1(x,y) = 8x^3 - x^2 + 4y^2$, $a_2(x,y) = 4y^2 - x^2 + 1$ and $\nu(\omega_1, \omega_2) = \begin{cases} 0, & \text{if } \omega_1 + \omega_2 \geq 2 \\ 2 - \omega_1 - \omega_2, & \text{if } \omega_1 + \omega_2 \leq 2 \end{cases}$. Then $b_t(x,y) = 8x^3 - x^2 + 4y^2 + t^2$.

1.4. THE GLUING THEOREM. A real polynomial a of two variables is called c o m p l e t e l y n o n - d e g e n e r a t e if it is peripherally non-degenerate and the curve $V_{(\mathbb{R}\setminus 0)^2}(a)$ is non-singular.

1.4.A. If $a_1, \dots, a_s$ are completely non-degenerate polynomials satisfying the conditions of 1.3 and if b_t are obtained by gluing of $a_1, \dots, a_s$ by some non-negative convex function satisfying the conditions (1), (2) and (3) of 1.3, then there exists $t_0 > 0$ such that for any $t \in (0, t_0]$ the polynomial b_t is completely non-degenerate and its chart can be obtained by gluing of charts of $a_1, \dots, a_s$.

A proof of this theorem and its generalization to high dimensions will be given in a separate paper [11]. Here we restrict ourselves to examples, discussions and applications of 1 4 A.

1.4.B.EXAMPLE. (Cf. 1.3.A and 1.2.A.) The polynomial $8x^3 - x^2 + 4y^2 + t^2$ with sufficiently small $t > 0$ has the chart shown in fig. 8.

1.5. BEHAVIOUR OF THE CURVE $V_{(\mathbb{R}\setminus 0)^2}(b_t)$ AS $t \to 0$. Let $a_1, \dots, a_s$, Δ, ν and b_t be as in 1.4.A and let $\nu|_{\Delta(a_1)} = 0$. According to 1.4.A the polynomial b_t with sufficiently small $t > 0$ has a chart obtained by gluing of charts of $a_1, \dots, a_s$. Obviously $b_0 = a_1$ since $\nu|_{\Delta(a_1)} = 0$. Thus when t passes to $t = 0$ the chart of a_1 stays only, the other charts disappear

How do domains containing the parts of $V_{(\mathbb{R}\setminus 0)^2}(b_t)$ homeomorphic to $V_{(\mathbb{R}\setminus 0)^2}(a_2), \dots, V_{(\mathbb{R}\setminus 0)^2}(a_s)$ behave as t becomes 0 ? They are going to the coordinate axes and to the infinity. The closer t to zero the more place is occupied by the domain where the curve $V_{(\mathbb{R}\setminus 0)^2}(b_t)$

is organized like $V_{(\mathbb{R}\setminus 0)^2}(a_1)$ and is approximated by it.

It is remarkable that the family b_t can be changed by simple geometric transformations in such a way that the role of a_1 passes to anyone of $a_2,\dots,a_s$ or even to a_k^Γ where Γ is a side of $\Delta(a_k)$, $k=1,\dots,s$. Indeed let $\lambda:\mathbb{R}^2\to\mathbb{R}$ be a linear function, $\lambda(x,y)=\alpha x+$ $+\beta y+\gamma$, and let $\nu'=\nu-\lambda$. Let b'_t be a result of gluing of $a_1,\dots,a_s$ by ν'. Let us denote a linear transformation $\mathbb{R}^2\to\mathbb{R}^2$: $:(x,y)\mapsto(t^a x, t^b y)$ by $qh_{(a,b),t}$. Then it is easy to verify that $V_{(\mathbb{R}\setminus 0)^2}(b'_t)=qh_{(\alpha,\beta),t}\,V_{(\mathbb{R}\setminus 0)^2}(b_t)$. However b'_t does not tend to a_1 as $t\to 0$. For example if $\lambda|_{\Delta(a_k)}=\nu|_{\Delta(a_k)}$ then $\nu'|_{\Delta(a_k)}=0$ and $b'_t\to a_k$. If the coincidence set of ν and λ is a side Γ of $\Delta(a_k)$ then $b'_t\to a_k^\Gamma$ and the curve $V_{(\mathbb{R}\setminus 0)^2}(b'_t)$ turns into $V_{(\mathbb{R}\setminus 0)^2}(a_k^\Gamma)$ as $t\to 0$.

The whole picture of change of $V_{(\mathbb{R}\setminus 0)^2}(b_t)$ when $t\to 0$ is the following. The fragments of $V_{(\mathbb{R}\setminus 0)^2}(b_t)$ organized as $V_{(\mathbb{R}\setminus 0)^2}(a_k)$ become more and more explicit. They are not staying, but are moving one from other. The only fragment that is growing without translation corresponds to the set where ν has its minimal value. The other fragments are moving away from it. Some of them (the ones going to the origin and to the axes) are contracting while the others are growing. But in the logarithmic coordinates (i.e. being transformed by $\ell:(x,y)\mapsto(\ln|x|,\ln|y|)$) all the fragments are growing. Changing ν we are applying linear transformations, which distinguish one fragment and cast away the others. The transformations turn our attention to a new piece of the curve. It is as if we transfer a magnifying lens from one fragment to another. Naturally under such magnification the other fragments disappear at the moment $t=0$.

1.6. GLUING AS REMOVING OF SINGULARITIES. In the projective plane $\mathbb{R}P^2$ the passage from curves defined by b_t with $t>0$ to the curve defined by b_0 looks quite differently. The domains where $V_{\mathbb{R}P^2}(b_t)$ is organized like the curves $V_{(\mathbb{R}\setminus 0)^2}(a_k)$ with $k=2,\dots,s$ are pressing to the points (1:0:0), (0:1:0), (0:0:1) and to the axes joining them. At $t=0$ they are as they were pressed into the points and axes.

Under the inverse passage (from $t=0$ to $t>0$) the full or partial removing of singularities concentrated at these points and lines takes place. It can be viewed also as a small perturbation of the polynomial $b_0=a_1$ defining the curve $V_{\mathbb{R}P^2}(b_0)$.

2. REMOVING OF SINGULARITIES OF TYPE J_{10}

2.1. CHARTS OF REMOVINGS. A singularity of a plane curve is said to be of type J_{10} if a germ of the curve at the singular point consists of three non-singular branches quadratically tangent to one other (see [1]). It is well-known (and follows from 1.1.A) that a curve $V_{\mathbb{R}^2}(a_1)$ has a singularity of the type J_{10} with real branches tangent to the axis of abscissas iff a_1 satisfy to the following three conditions: (1) only one side of $\Delta(a_1)$ is turned toward to the origin, (2) this side is the segment Γ with the end-points (6,0) and (0,3), (3) $a_1^\Gamma(x,y)=\alpha(y-\alpha_1 x^2)(y-\alpha_2 x^2)(y-\alpha_3 x^2)$, where $\alpha,\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$ and $\alpha_i\neq\alpha_j$ for $i\neq j$.

By virtue of 1.4.A for removing of a such singularity of $V_{\mathbb{R}^2}(a_1)$ it is sufficient to glue a completely non-degenerate polynomial a_2 with $\Delta(a_2)=$  and $a_2^\Gamma=a_1^\Gamma$ to the polynomial a_1. The chart of a_2 defines the topology of a curve obtained, therefore it will be called a chart of the removing. The following theorem gives a set of removings of a singularity of the type J_{10} with 31 topologically distinct charts. As it will be proved in §3 this set is complete.

2.1.A. For any $\alpha_1,\alpha_2,\alpha_3>0$ distinct from one other there exist completely non-degenerate polynomials having the Newton polygon , Γ-truncation $(y-\alpha_1 x^2)(y-\alpha_2 x^2)(y-\alpha_3 x^2)$ and the charts shown in fig. 9.

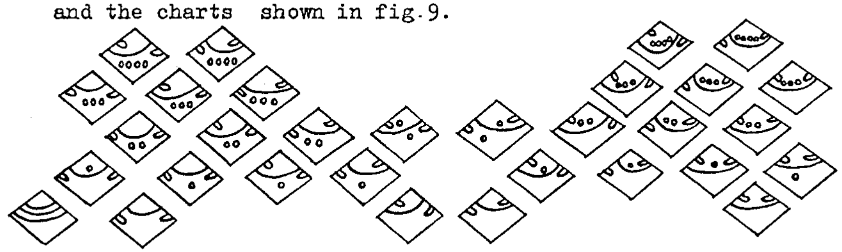
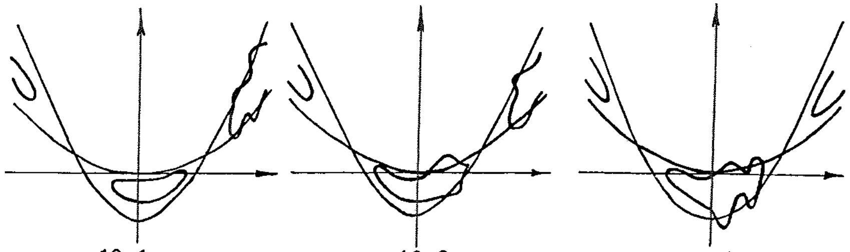


Fig. 9.

2.2. LEMMA. For any $\beta_0>\beta_1>\beta_2>\beta_3>0$ with $\beta_0+\beta_3=\beta_1+\beta_2$ and for any $j=1,2,3$ there exists a completely non-degenerate polynomial b_j such that (i) $\Delta(b)$ is the triangle  bounded by segments of coordinate axes and the segment Σ with the end-point (4,0) and (0,2), (ii) $b_j^\Sigma(x,y)=(y-\beta_1 x^2)(y-\beta_2 x^2)$, (iii) the curve $V_{\mathbb{R}^2}(b)$ disposes with respect to the paraboles $V_{\mathbb{R}^2}(y-\beta_0 x^2+1)$ and $V_{\mathbb{R}^2}(y-\beta_s x^2)$ as is shown in fig 10.j.

PROOF. Let us denote the polynomials $y-\beta_0 x^2+1$ and $y-\beta_3 x^2$ by ρ_0 and ρ_3. Set $l_i(x,y)=x-\gamma_i$ with $i=1,2,3,4$ and $c_t=\rho_0\rho_3+$

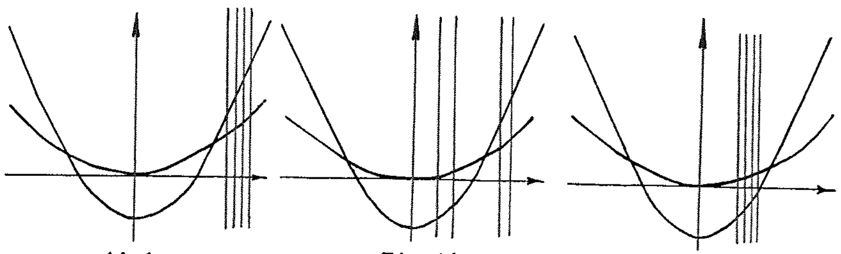
$+t\,l_1 l_2 l_3 l_4$. It is clear that $C_t^{\Sigma}(x,y)=(y-\beta_0x^2)(y-\beta_3x^2)+tx^4$. On the other hand C_t^{Σ} can be decomposed: $C_t^{\Sigma}(x,y)=(y-\delta_1x^2)(y-\delta_2x^2)$. So $\delta_1+\delta_2=\beta_0+\beta_3$ and $\delta_1\delta_2=\beta_0\beta_3+t$. Since $\beta_0+\beta_3=\beta_1+\beta_2$ and $\beta_0>\beta_1>\beta_2>\beta_3>0$ then $\beta_1\beta_2>\beta_0\beta_3$ and for $t=\beta_1\beta_2-\beta_0\beta_3$ we have $C_t^{\Sigma}(x,y)=(y-\beta_1x^2)(y-\beta_2x^2)$. Thus $C_{\beta_1\beta_2-\beta_0\beta_3}$ satisfies (i) and (ii) independently on the choice of $\gamma_1,\dots,\gamma_4$. Let us show that $\gamma_1,\dots,\gamma_4$ can be chosen so that the condition (iii) be satisfied.



10.1 10.2 10.3

Fig.10

If the lines $V_{\mathbb{R}^2}(l_i)$ are situated with respect to $V_{\mathbb{R}^2}(p_k)$ as in fig.11.j, then there exists $\varepsilon>0$ such that for $t\in(0,\varepsilon]$ the curve $V_{\mathbb{R}^2}(C_t)$ consists of 3 components and is disposed with respect to $V_{\mathbb{R}^2}(p_k)$ as in fig.10.j. Let us show that by a choice



11.1 Fig.11 11.3

of l_i we can achieve that the role of ε can be played by any number from the interval $(0,(\beta_0^2+\beta_3^2)/2)$ and in particular, $\beta_1\beta_2-\beta_0\beta_3$. As $\mathrm{Int}\,\Delta(C_t)$ contains only one integer point, the genus of the curve determined by C_t is not more than 1. Therefore under the increasing of t the first modification of $V_{\mathbb{R}^2}(C_t)$ with $t>0$ must either diminish the number of components or give a decomposing curve.

The latter is impossible. In fact consideration of C^{Σ} shows that curves constituting $V_{\mathbb{R}^2}(C_t)$ are to be either two imagine

conjugate curves or two real parabolas. The first is impossible since for $t>0$ any line $V_{\mathbb{R}^2}(x^2-\gamma)$ with $\gamma\in(\gamma_1,\gamma_2)$ intersects $V_{\mathbb{R}^2}(C_t)$ in two real points, the second is impossible since a vertical line passing through a point of $V_{\mathbb{R}^2}(p_0)\cap V_{\mathbb{R}^2}(p_3)$ does not intersect $V_{\mathbb{R}^2}(C_t)$ for $t>0$.

For $t\in(0,(\beta_0^2+\beta_3^2)/2)$ there exist branches of $V_{\mathbb{R}^2}(C_t)$ going to infinity. Choosing lines $V_{\mathbb{R}^2}(l_i)$ sufficiently close to one other and to a point of $V_{\mathbb{R}^2}(p_0)\cap V_{\mathbb{R}^2}(p_3)$ we can achieve that for any $t\in(0,(\beta_0^2+\beta_3^2)/2)$ there exist two branchs of $V_{\mathbb{R}^2}(C_t)$ in some neighbourhood of this point and therefore $V_{\mathbb{R}^2}(C_t)$ has 3 components i.e. there is no modification.

2.3. PROOF OF 2.1.A. The chart shown in the left lower corner of fig.9 is the chart of $(y-\alpha_1(x^2+1))(y-\alpha_2(x^2+1))(y-\alpha_3(x^2+1))$. The other charts of fig.9 are realized by polynomials obtained by small perturbings of $p_k b_j$, where p_{k_s} and b_j are as in 2.2. The perturbings are to be made by adding $\varepsilon\prod_{i=1}(x-\delta_i)$. It does not change the monomials corresponding to the points of Γ.

By perturbing of $p_3 b_j$ we obtain polynomials with any desired Γ-truncation since in 2.3 the set of β_1,β_2,β_3 subjects only to the restriction $\beta_1>\beta_2>\beta_3>0$. That is not so in the case of $p_0 b_j$. In 2.2 the set $\{\beta_0,\beta_1,\beta_2\}$ subjects to the restriction $\beta_1+\beta_2-\beta_0>0$ since $\beta_1+\beta_2-\beta_0=\beta_3>0$. Therefore by perturbing of $p_0 b_j$ we obtain polynomials with Γ-truncations $(y-\alpha_1x^2)(y-\alpha_2x^2)(y-\alpha_3x^2)$ where $\alpha_2+\alpha_3-\alpha_1>0$ and $\alpha_1>\alpha_2>\alpha_3>0$. To obtain the polynomials with arbitrary $\alpha_1>\alpha_2>\alpha_3>0$ we choose δ such that the numbers $\alpha_i'=\alpha_i+\delta$ satisfy the restriction $\alpha_2'+\alpha_3'-\alpha_1'>0$, construct the desired polynomials with Γ-truncation $(y-\alpha_1'x^2)(y-\alpha_2'x^2)(y-\alpha_3'x^2)$ and apply the transformation $(x,y)\mapsto(x,y+\delta x^2)$. ▨

3. CONSTRUCTING NON-SINGULAR CURVES OF DEGREES 6 AND 7

3.1. METHOD OF DESCRIBING THE ISOTOPY TYPE OF A NON-SINGULAR CURVE. The isotopy type of a non-singular curve of degree m is determined by the scheme of nutual disposition of its components (it is also called its real scheme). For description of the real schemes we shall apply the following system of notation.

A curve consisting of one oval is encoded by the symbol $\langle 1\rangle$, the empty curve by the symbol $\langle 0\rangle$, a connected one-sided curve by the symbol $\langle J\rangle$. If the symbol $\langle A\rangle$ encodes some set of ovals then the set obtained from it by adjoining one oval enclosing all the rest is encoded by the symbol $\langle 1\langle A\rangle\rangle$. A curve presented as the union of two nonintersecting curves, which are encoded by the symbols $\langle A\rangle$ and $\langle B\rangle$ and such that no oval of one curve is enclosed by an oval of the other, is encoded by the symbol $\langle A \amalg B\rangle$. We shall use two abbreviations: first, if $\langle A\rangle$ is the code of a set of ovals, then a fragment of another code having the form $A \amalg \ldots \amalg A$, where A is repeated n times, is abbreviated by $n \times A$; second, fragements of a code having the form $n \times 1$ are abbreviated by the notation n .

3.2. CURVES OF DEGREE 6. The following theorem on isotopy classification of non-singular plane projective real algebraic curves of degree 6 was proved by D.A.Gudkov [3] in the late sixties.

3.2.A. There exist nonsingular curves of degree 6 with the following real schemes:

(i) $\langle 9 \amalg 1\langle 1\rangle\rangle$, $\langle 5 \amalg 1\langle 5\rangle\rangle$, $\langle 1 \amalg 1\langle 9\rangle\rangle$;

(ii) $\langle 10\rangle$, $\langle 8 \amalg 1\langle 1\rangle\rangle$, $\langle 5 \amalg 1\langle 4\rangle\rangle$, $\langle 4 \amalg 1\langle 5\rangle\rangle$, $\langle 1 \amalg 1\langle 8\rangle\rangle$, $\langle 1\langle 9\rangle\rangle$;

(iii) $\langle \alpha \amalg 1\langle\beta\rangle\rangle$ with $\alpha+\beta \leqslant 8$, $0 \leqslant \alpha \leqslant 7$, $1 \leqslant \beta \leqslant 8$;

(iv) $\langle\alpha\rangle$ with $0 \leqslant \alpha \leqslant 9$; (v) $\langle 1\langle 1\langle 1\rangle\rangle\rangle$.

Any nonsingular curve of degree 6 has one of these 56 real schemes.

All these schemes, except for schemes $\langle 5 \amalg 1\langle 5\rangle\rangle$ and $\langle 4 \amalg 1\langle 5\rangle\rangle$, are realized by the classic methods of Harnack and Hilbert, see Gudkov [3] . Hilbert conjectured that $\langle 5 \amalg 1\langle 5\rangle\rangle$ is unrealizable. Gudkov [3] , refuted this conjecture by a very complicated construction. The following construction based on 1.4.A and 2.1 A seems to be simpler than Gudkov's one Moreover it gives a realization of all 55 nonempty schemes of Theorem 3 2 A.

3.2.B. CONSTRUCTION. Apply to the polynomials of 2.1.A the transformation $a(x,y) \mapsto y^6 a\left(\frac{x}{y}, \frac{1}{y}\right)$. The charts of the results are shown in fig 12. By gluing of these polynomials and the polynomials

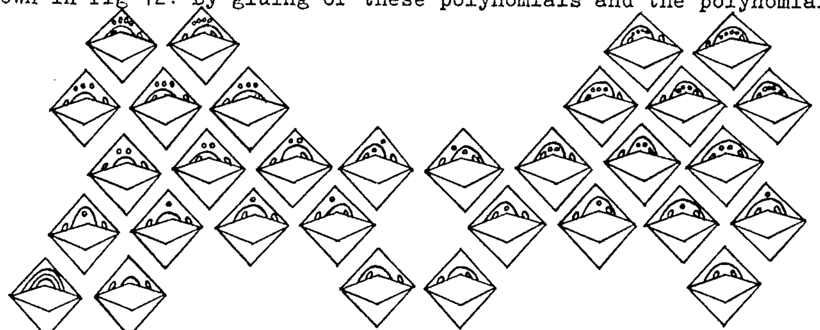


Fig. 12

of 2.1.A we obtain the polynomials defining nonsingular curves of degree 6 with all 55 desired real schemes of 3.2.A.

The gluing may be considered as the gluing the polynomials above to $(y-\alpha_1 x^2)(y-\alpha_2 x^2)(y-\alpha_3 x^2)$. The curve $V_{\mathbb{R}P^2}((y-\alpha_1 x^2)(y-\alpha_2 x^2) \times (y-\alpha_3 x^2))$ is the union of 3 conics. It has 2 singular points, each of which is of the type J_{10} . Thus nonsingular curves of degree 6 with all realizable non-empty real schemes can be obtained by small perturbations of the curve $V_{\mathbb{R}P^2}((y-\alpha_1 x^2)(y-\alpha_2 x^2)(y-\alpha_3 x^2))$. The latter is shown in fig 13. In fig. 14 it is shown a curve obtained from it with the real scheme $\langle 5 \amalg 1\langle 5\rangle\rangle$.

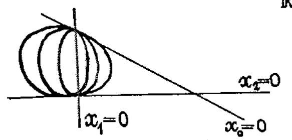


Fig. 13.

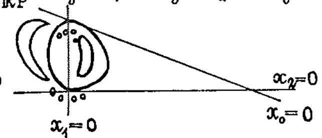


Fig. 14

As a consequence of this construction and the isotopy classification of the non-singular curves of degree 6, we obtain the completeness of the set of removing charts for J_{10} shown in fig 9.

3.2.C. Any completely non-degenerate polynomial having the Newton polygon  and Γ-truncation $(y-\alpha_1 x^2)(y-x_2 x^2)(y-\alpha_3 x^2)$ with $\alpha_1, \alpha_2, \alpha_3 > 0$ has one of the charts shown in fig 9.

PROOF. The gluing of the chart of the polynomial in question to any chart of fig.12 must give a scheme of 3.2.A. It can be easily verified that the only schemes having this property are the schemes of fig.9.

Another way of proving 3 2.C is provided by [9] . ∎

3.3. CURVES OF DEGREE 7. The following theorem on isotopy classification of the non-singular plane projective real algebraic curves of degree 7 was announced in my article [8] .

3.3.A. There exist nonsingular curves of degree 7 with the following real schemes:

(i) $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ with $\alpha+\beta \leqslant 14$, $0 \leqslant \alpha \leqslant 13$, $1 \leqslant \beta \leqslant 13$;

(ii) $\langle J \amalg \alpha\rangle$ with $0 \leqslant \alpha \leqslant 15$; (iii) $\langle J \amalg 1\langle 1\langle 1\rangle\rangle\rangle$.

Any nosingular curve of degree 7 has one of these 121 real schemes.

Up to the time the work [8] was being dome it remained unknown whether there exist curves with the schemes $\langle J \amalg 1\langle 14\rangle\rangle$, $\langle J \amalg 10 \amalg 1\langle 4\rangle\rangle$ and $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$ with $13 \leqslant \alpha+\beta \leqslant 14$, $3 \leqslant \alpha$, $6 \leqslant \beta$.

In fact, an application of the Bezout theorem to the curve in question and some auxiliary curves of degree 1 and 2 shows that a real scheme of a nonsingular curve of degree 7 belongs to one of the following types: $\langle J \amalg \alpha\rangle$, $\langle J \amalg \alpha \amalg 1\langle\beta\rangle\rangle$, $\langle J \amalg 1\langle 1\langle 1\rangle\rangle\rangle$, where $\alpha \geqslant 0$,

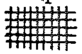
$\beta \geq 1$. By the Harnack inequality the number of components of a nonsingular curve of degree 7 is not more than 16. By the Harnack method the following real schemes are realized: $\langle J \amalg \alpha \rangle$ with $0 \leq \alpha \leq 15$; $\langle J \amalg \alpha \amalg \langle 1 \rangle\rangle$ with $0 \leq \alpha \leq 13$ and $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle\rangle$ with $0 \leq \alpha \leq 9$, $0 \leq \beta \leq 4$. By the Hilbert method the following real schemes are realized: $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle\rangle$ with $\alpha + \beta \leq 12$; $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle\rangle$ with $\alpha + \beta \leq 14$, $\alpha \leq 2$, $\beta \leq 13$ and $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle\rangle$ with $\alpha + \beta \leq 14$, $\alpha \leq 12$, $\beta \leq 3$. Gudkov's construction [4] gives the real schemes $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle\rangle$ with $\alpha \leq 9$, $\beta \leq 5$.

Thus Theorem 3.3.A is reduced to the following two theorems

3.3.B. There does not exist a nonsingular curve of degree 7 with the real scheme $\langle J \amalg 1 \langle 14 \rangle\rangle$.

3.3.C. There exist nonsingular curves of degree 7 with the real schemes $\langle J \amalg \alpha \amalg 1 \langle \beta \rangle\rangle$ with $6 \leq \alpha + \beta \leq 14$, $1 \leq \alpha$, $2 \leq \beta$.

Theorem 3.3.B is proved in my article [10]. Theorem 3.3.C is proved in the rest of this article.

3.3.D. LEMMA. There exist 4 completely non-degenerate polynomials with the Newton polygon  and the charts shown in fig.15.

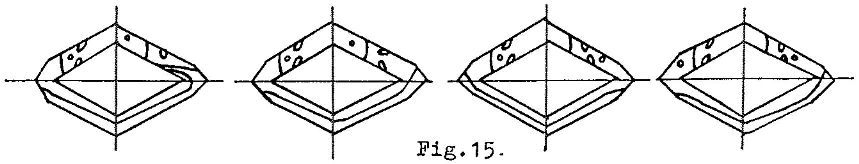


Fig.15.

In other words there exist 4 curves of degree 7 disposed as is shown in fig 16 Each of the curves has two singular points of the type J_{10}.

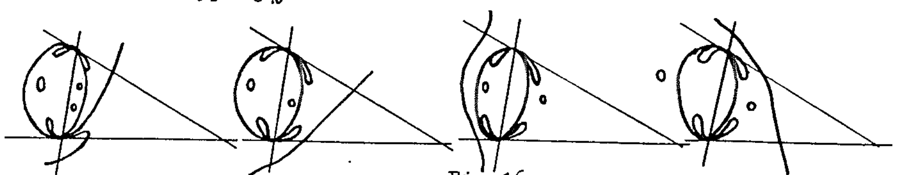


Fig.16.

PROOF. Let us use the Hilbert method adapted to construction of singular curves. By small perturbings of the unions of $V_{\mathbb{R}P^2}(y-x^2)$ and the straight lines shown in fig.17 let us construct 4 nonsingular curves of degree 3 disposed with respect to $V_{\mathbb{R}P^2}(y-x^2)$ and the coordinate axes as is shown in fig.18

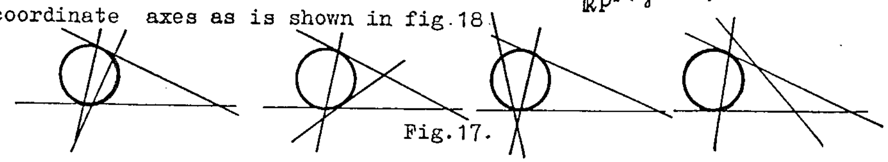


Fig.17.

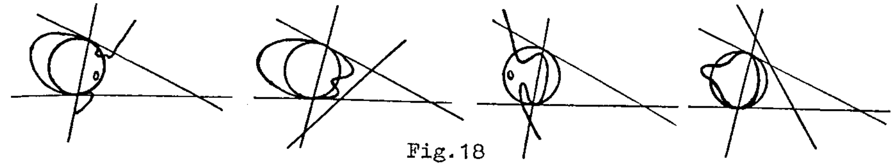


Fig.18

Perturb the unions of the curves obtained and $V_{\mathbb{R}P^2}(y-x^2)$ to obtain the curves of degree 5 shown in fig 19.

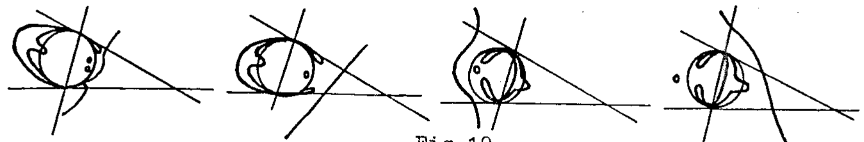


Fig.19

The unions of the curves of degree 5 obtained and $V_{\mathbb{R}P^2}(y-x)^2$ can be obviously perturdeb so that the desired curves be obtained. ▨

PROOF OF 3.3.C. Remove the singularities of the curves of fig 16. Do this by gluing the polynomials of Lemma 3.3.D with the polynomials having the charts shown in fig.9 and fig.12.

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