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Hence $c_i = 0$ and $d_i = 0$ for all i.

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2.43 **Theorem.** For any subspaces U_1 and U_2 of a finit-dimensional space,

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Proof. Let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$.
Extend it to a basis $u_1, \ldots, u_m, v_1, \ldots, v_j$ of U_1,
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Clearly, $U_1, U_2 \subset \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$.
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Let us prove that the list $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is linearly indepedent.

Let $a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_k w_k = 0$.
Rewrite as $c_1 w_1 + \cdots + c_k w_k = -a_1 u_1 - \cdots - a_m u_m - b_1 v_1 - \cdots - b_j v_j$.
Hence $c_1 w_1 + \cdots + c_k w_k \in U_1 \cap U_2$, and $c_1 w_1 + \cdots + c_k w_k = d_1 u_1 + \cdots + d_m u_m$.
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