Lecture 1

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This course, MAT 150 Introduction to Advanced Mathematics, has two goals. First, it teaches the language of mathematics: basic logic and set theory. Second, it contains a collection of small stories from various areas of mathematics in an attempt to answer to the question **What is the mathematics that mathematicians like?** Indeed, mathematicians like the mathematics that most people are not aware about.

No preliminary knowledge of advanced mathematics is required. No textbook is mandatory.

The world of numbers and the likes. Mathematics studies objects, like numbers, which do belong neither to the physical world, nor to the world of social relations. They form their own world. We will call it a *mathematical universe*. Its objects can be modelled in many ways in the physical world, but the physical world is not their natural habitat.

On the other hand, sooner or later a study of anything starts to use mathematics. In its mature stage, any science uses mathematics, looks at its own objects through the glass of mathematical models.

Nonetheless, mathematics is not just a tool set of models for other sciences. Mathematical universe is a real consistent world. Any attempt to divide it to unrelated pieces fails. It deserves to be studied both for its usefulness and beauty.

The language for mathematics.

> In the beginning was the Word,
> and the Word was with God,
> and the Word was God.
>
> The Bible

The mathematical universe is not open directly to human senses. Any study of mathematics, any penetration to the mathematical universe, starts with words.

Introductory stories about the mathematical universe are delivered in a common language (no matter which), but soon the language starts to change, to develop, because it has to adjust to mathematical realities.

A similar story happens to a language used by any other science. Each science works out specific notions, and they require new words. Adding one by one these extra words converts a common language into a jargon specific to the science.

The mathematical language goes far beyond this. Mathematics requires a new layer of language which is necessary for proofs. To a great extent, proofs in mathematics take the role of human senses and devices that enhance the senses.

Here the language of proofs has to be understood in a broad sense. It includes the system of notions and constructions for making definitions and statements, as well as words necessary for adequate presentations of mathematical objects. All in all, this stuff forms a part of mathematics.

Why different and why difficult. For a student, the mathematical language is a challenge to learn. At first glance, it is not needed. The most important words in the mathematical language are borrowed from the common vocabulary. Apparently, mathematicians speak the same plain English (sometimes broken) as everybody else. The differences are subtle, almost invisible for a non-mathematician.

A mathematician is used to speak a mathematically meaningful language, so that the difference is not noticed by a mathematician, either. It is not noticed until a non-mathematician tries to rephrase what was said. A mathematician may be surprised by the change of meaning.

The most profound and common difficulty in studying of mathematics comes from poor skills in mathematical language. In the low level mathematical courses (up to Calculus and Linear Algebra) the language is not that important, because these courses are targeted at solving of standard problems. Proofs are avoided. The success in any mathematics course above Calculus depends first of all on understanding of the mathematical language.

One can easily see this looking at grades. As soon as a student gets the language, all the mathematical courses become equally easy. The grades are uniform across the subjects and do not depend even on student's efforts.

In this course, you will learn the basics of the mathematical language.

Our path to mastering language. We will build the mathematical language on top of the natural one, the plain English. When one studies another natural language, the very first task is to learn words. Contrary to this, we will use common English words, but discuss one by one those words which have specific meaning in the mathematical language. Usually, we will consider whole groups of words, discuss synonyms and versions of usage.

The order in which we will study words is quite unusual. Usually, when one starts to learn a foreign language, the bulk of new words that are learned on the first stage are nouns and verbs. In our study of the mathematical language, after study of two groups of nouns and verbs, we will dive into a detailed study of conjunctions.

1 Sets

Why sets? In any intellectual activity, one of the most profound actions is gathering objects in groups. The gathering is performed in mind and is not supposed to be accompanied with any action in the physical world. As soon as a group has been created and assigned a name, it can be a subject of thoughts and arguments and, in particular, it can be included into other groups.

Mathematics has an elaborated system of notions, which organizes and regulates creating those groups and manipulating them. The system is called the *naive set theory*. This name is slightly misleading, because this is rather a language than a theory.

The first words. The first words in the mathematical language are *set* and *element*. By a *set* we understand an arbitrary collection of various objects. An object included into the collection is called an *element* of the set.

A few words are used for the relation between a set and its elements. An element *belongs* to a set. A set *contains* an element. A set *consists* of its elements. Elements *form* the set.

In order to diversify the wording, the word *set* is replaced by the word *collection*. Sometimes other words, such as *class*, *family*, and *group*, are used in the same sense, but this is not quite safe, because each of these words is associated in modern mathematics with a more special meaning, and hence should be used instead of the word *set* with a caution.

The first formulas. If x is an element of a set A, then we write $x \in A$.

The sign $\in$ is a version of the Greek letter epsilon, which corresponds to the first letter of the Latin word *element*. To make the notation more flexible, the formula $x \in A$ is also allowed to be written backwards, that is in the form $A \ni x$.

This disrespect to the origin of the notation is payed off by emphasizing a meaningful similarity of $\in$ and $\ni$ to the inequality symbols $<$ and $>$.

To state that x is not an element of A, we write $x \notin A$ or $A \not\ni x$.

Equality of sets. A set is determined by its elements. The set is nothing but a collection of its elements. This manifests most sharply in the following principle (called *Axiom of Extensionality*):

Two sets are considered equal if and only if they have the same elements.

In this sense, the word *set* has slightly disparaging meaning. When something is called a set, this shows, maybe unintentionally, a lack of interest to whatever organization of the elements of this set.

For example, when one says that a line is a set of points, it implies that two lines coincide if and only if they consist of the same points. On the other hand, it means that all relations between points on a line (e.g., the distance between points, the order of points on the line, etc.) are not included into the notion of line.

Built in a commitment to take them easy. You may think of sets as of boxes that can be built effortlessly around elements, just to distinguish them from the rest of the world. The cost of this lightness is that such a box is not more than the collection of elements placed inside.

This is a little more than just a name: it's a declaration of our intention to think about this collection of things as of an entity and not to go into details about the nature of its members-elements. Elements, in turn, may also be sets, but as long as we consider them elements, they play the role of atoms, with their own original nature ignored.

Overuse of the words. In modern Mathematics, the words *set* and *element* are very common and appear in most texts. They are even overused, that is they are used at instances when it is not appropriate.

For example, it is not good to use the word *element* as a replacement for other, more meaningful word.

> When you call something an *element*, then the *set,*
> whose element this one is, should be clear.

The word *element* makes sense only in combination with the word *set*, unless we deal with a non-mathematical term (like a *chemical element*), or a rare old-fashioned exception from the common mathematical terminology (sometimes the expression under the sign of integral is called an *infinitesimal element*; lines, planes, and other geometric images are also called *elements* in old texts). Euclid's famous book on Geometry is called *Elements*, too.

The empty set. Thus, an element may not be without a set. However, a set may have no elements. Actually, there is such a set. This set is unique, because a set is completely determined by its elements. It is called the *empty set* denoted[1] by $\varnothing$.

Basic sets of numbers. In addition to $\varnothing$, there are some other sets so important that they have their own special names and denoted by special symbols.

- The set of all positive integers, i.e., 1, 2, 3, 4,. . . , etc., is denoted by $\mathbb{N}$.
- The set of all integers, both positive, negative, and the zero, is denoted by $\mathbb{Z}$.
- The set of all rational numbers (join to the integers all the numbers that are presented by fractions, like $2/3$ and $\frac{-7}{5}$) is denoted by $\mathbb{Q}$.
- The set of all real numbers (obtained by adjoining to rational numbers the numbers like $\sqrt{2}$ and $\pi = 3.14\dots$) is denoted by $\mathbb{R}$.
- The set of all complex numbers is denoted by $\mathbb{C}$.

[1]Other symbols, like Λ, are also in use, but $\varnothing$ has become most common one.

Describing a set by a list of its elements. A set presented by a list a, b, $\dots$, x of its elements is denoted by the symbol $\{a, b, \dots, x\}$. In other words, the list of objects enclosed in curly brackets denotes the set whose elements are listed. For example, $\{1, 2, 123\}$ denotes the set consisting of the numbers 1, 2, and 123. The symbol $\{a, x, A\}$ denotes the set consisting of three elements: a, x, and A, whatever objects these three letters denote.

In order to check whether you take this correctly, please, answer to the following questions.

1.1. Is it true that $\varnothing = \{\varnothing\}$?

1.2. What is $\{\varnothing\}$? How many elements does it contain?

1.3. Which of the following formulas are correct:
1) $\varnothing \in \{\varnothing, \{\varnothing\}\}$; 2) $\{\varnothing\} \in \{\{\varnothing\}\}$; 3) $\varnothing \in \{\{\varnothing\}\}$?

A set consisting of a single element is called a *singleton*. This is any set which can be presented as $\{a\}$ for some a.

1.4. Is $\{\{\varnothing\}\}$ a singleton?

1.5. Is it true that $\{1, 2, 3\} = \{3, 2, 1, 2\}$?

At first glance, lists with repetitions are never needed. There even arises a temptation to prohibit usage of lists with repetitions in such notation. However, as it often happens to temptations to prohibit something, this would not be wise. Indeed, quite often one cannot say a priori whether there are repetitions or not. For example, the elements in the list may depend on a parameter, and under certain values of the parameter some entries of the list coincide, while for other values they don't.

1.6. How many elements do the following sets contain?
1) $\{1, 2, 1\}$; 2) $\{1, 2, \{1, 2\}\}$; 3) $\{\{2\}\}$;
4) $\{\{1\}, 1\}$; 5) $\{1, \varnothing\}$; 6) $\{\{\varnothing\}, \varnothing\}$;
7) $\{\{\varnothing\}, \{\varnothing\}\}$; 8) $\{x, 3x - 1\}$, where $x \in \mathbb{R}$.

2 Maps

In modern mathematics, objects are always communicating to each other. Sets communicate to each other via maps.

A *map* f of a set X to a set Y is a triple consisting of X, Y, and a rule, which assigns to every element of X exactly one element of Y.

There are other words with the same meaning: *mapping*, *function*, etc. (Special kinds of maps may have special names like *functional*, *operator*, *sequence*, *family*, *fibration*, etc.)

If f is a map of X to Y, then we write $f : X \to Y$, or $X \xrightarrow{f} Y$. The element b of Y assigned by f to an element a of X is denoted by $f(a)$ and called the *image* of a under f, or the f-*image* of a. In order to state that $b = f(a)$, one may write also $a \overset{f}{\mapsto} b$, or $f : a \mapsto b$. We also define maps by formulas like $f : X \to Y : a \mapsto b$, where b is explicitly expressed in terms of a.

> **2.1.** Let X and Y be sets consisting of p and q elements, respectively. Find the number of maps $X \to Y$.

Identity The *identity map* of a set X is the map $\mathrm{id}_X : X \to X : x \mapsto x$. It is denoted by id if X is clear from the context.

Compositions The *composition* of maps $f : X \to Y$ and $g : Y \to Z$ is the map $g \circ f : X \to Z : x \mapsto g\big(f(x)\big)$.

2.A (Associativity)**.** $h \circ (g \circ f) = (h \circ g) \circ f$ for any maps $f : X \to Y$, $g : Y \to Z$, and $h : Z \to U$.

Proof. Let $x \in X$. Then

$$h \circ (g \circ f)(x) = h(g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = (h \circ g) \circ f(x).$$

□

2.B. $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for any $f : X \to Y$.

3 Subsets and inclusions.

If A and B are sets and every element of A also belongs to B, then we say that A is a *subset* of B, or B *includes* or *contains* A, and write $A \subset B$ or $B \supset A$.

The inclusion signs $\subset$ and $\supset$ resemble the inequality signs $<$ and $>$, but not quite: no number a satisfies the inequality $a < a$, while any set A contains itself:

3.A (Reflexivity of inclusion). *Inclusion $A \subset A$ holds true for any A.*

Proof. Recall that, by the definition of an inclusion, $A \subset B$ means that each element of A is an element of B. Therefore, the statement that we must prove can be rephrased as follows: each element of A is an element of A. This is tautologically correct. □

Thus, the inclusion signs are not like inequality signs $<$ and $>$. They are closer to $\leq$ and $\geq$.

Sometimes, being inspired by signs $\leq$ and $\geq$, inclusions are denoted by symbols $\subseteq$ and $\supseteq$ or even $\subseteqq$ and $\supseteqq$, reserving the symbols $\subset$ and $\supset$ for strict inclusions, that prohibit equality, like strict inequalities. We follow the mainstream mathematical notation in which the signs $\subseteq$, $\supseteq$, $\subseteqq$ and $\supseteqq$ are not used and strict inclusions are denoted by $\subsetneq$ and $\supsetneq$ or by $\underset{\neq}{\subset}$ and $\underset{\neq}{\supset}$.

3.B (Ubiguity of the empty set). $\varnothing \subset A$ *for any set* A*. In other words, the empty set is present in each set as a subset.*

Proof. Recall that, by the definition of inclusion, $A \subset B$ means that each element of A is an element of B. Thus, we need to prove that any element of $\varnothing$ belongs to A. This is true because $\varnothing$ does not contain any element. □

It may happen that you are not satisfied with this proof. Arguments about the empty set may confuse at first. To this end, look at

Another proof of 3.B. Let us resort to the question whether the statement which we prove can be wrong. How can it happen that $\varnothing$ is not a subset of A? This is possible only if $\varnothing$ contains an element which is not an element of A. However, $\varnothing$ does not contain such elements because $\varnothing$ contains no elements at all. □

Thus, each set A has two obvious subsets: the empty set $\varnothing$ and A itself. A subset of A different from $\varnothing$ and A is called a *proper* subset of A. This word is used when we do not want to consider the obvious subsets (which are *improper*).

3.C (Transitivity of inclusion). *If* A*,* B*, and* C *are sets,* $A \subset B$ *and* $B \subset C$*, then* $A \subset C$*.*

Proof. We must prove that each element of A is an element of C. Let $x \in A$. Since $A \subset B$, it follows that $x \in B$. Since $B \subset C$, the latter (i.e., $x \in B$) implies $x \in C$. This is what we had to prove. □

Defining a set by a condition (a set-builder notation) As we know, a set can be described by presenting a list of its elements. This simplest way may be not available or, at least, be not the easiest one. For example, it is easy to say: "the set of all solutions of the following equation" and write down the equation. This is a reasonable description of the set. At least, it is unambiguous. Having accepted it, we may start speaking on the set, studying its properties, and eventually may be lucky to solve the equation and obtain the list of its solutions. Although the latter task may be difficult, this should not prevent us from discussing the set until the time when the equation will be solved. (Solution of some equations took centuries!)

Thus, we see another way for describing a set: formulate properties that distinguish the elements of the set among elements of some wider and already known set. Here is the corresponding notation:

The subset of a set A consisting of the elements x that satisfy a condition $P(x)$ is denoted by $\{x \in A \mid P(x)\}$.

3.1. Present the following sets by lists of their elements (i.e., in the form $\{a, b, \dots\}$)
(a) $\{x \in \mathbb{N} \mid x < 5\}$, (b) $\{x \in \mathbb{N} \mid x < 0\}$, (c) $\{x \in \mathbb{Z} \mid x < 0\}$.

The set-builder notation unveils a close relation between logic statements and sets. Every statement P about elements of a set A defines a subset $\{x \in A \mid P(x)\}$ of A. On the other hand, any subset $B \subset A$ gives rise to a property of elements of A: namely, the property of belonging to B, that is $x \in B$.

For example, let us figure out what on the side of logic statements corresponds to inclusion. Let B and C are subsets of a set A. Let $B = \{x \in A \mid P(x)\}$ and $C = \{x \in A \mid Q(x)\}$, that is P and Q are the statements defining B and C, respectively. Inclusion $B \subset C$ means that each element of B is an element of C. In other words, if $x \in B$, then $x \in C$, or, in terms of P and Q, if $P(x)$, then $Q(x)$. Thus, the inclusion $B \subset C$ corresponds to implication "if $P(x)$, then $Q(x)$".

4 Predicates

Denote by $\mathcal{B}$ the set which consists of two elements, *true* and *false*. We will abbreviate these long names to T and F, respectively. Here the symbol $\mathcal{B}$ is for George Boole (1815-1864).

Let U be a set. A map $P : U \to \mathcal{B}$ is called a ***predicate***. One can think about a predicate as about a property of elements of the set U. An element $x \in U$ has the property if $P(x) = true$ and has no the property if $P(x) = false$. A predicate P defines the set $\{x \in U \mid P(x) = true\}$ of elements that have the property.