page 39 #5

In (a), (b), and (c) the sets can be expressed as

$$\prod_{i\in\mathbb{N}} A_i$$

where

(a) $A_i = \mathbb{Z}$

(b) $A_i = [i, \infty)$

(c) $A_i = \begin{cases} \mathbb{R} & \text{for } i < 100 \\ \mathbb{Z} & \text{for } i \geq 100 \end{cases}.$

In (d) this is impossible as follows: the set contains $(0,0,0,0,\ldots)$ and $(0,1,1,0,0,0,\ldots)$, so A_2 and A_3 must each contain 0 and 1, but then the product must contain $(0,0,1,0,0,0,\ldots)$ which is a contradiction.

page 83 #3

$\emptyset \in \mathcal{T}_c$ since $X - \emptyset$ is all of X. $X \in \mathcal{T}_c$ since $X - X = \emptyset$ which is countable.

Let $\{U_\alpha\}$ be an indexed family of elements of $\mathcal{T}_c$. WLOG assume they are non-empty. $X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$ which is an intersection of countable sets so it is countable. Thus arbitray unions of elements of $\mathcal{T}_c$ are in $\mathcal{T}_c$.

Let $U_1, \ldots, U_n$ be elements of $\mathcal{T}_c$. If any of them are empty then their intesection is also empty so it is in $\mathcal{T}_c$. Otherwise, $X - \bigcap U_i = \bigcup (X - U_i)$ is a finite union of countable sets so it is countable. Thus finite intersections of $\mathcal{T}_c$ are in $\mathcal{T}_c$.

$\mathcal{T}_\infty$ is not a topology on $\mathbb{Z}$ because

$$\bigcup_{i\neq 0} \{i\}$$

is a union of elements in $\mathcal{T}_\infty$ but its complement is $\{0\}$ which is finite, nonempty, and not all of $\mathbb{Z}$.

page 83 #4

(a) Each $\mathcal{T}_\alpha$ contains the empty set and the whole space since they are topologies so their intersection does too. Let $\{U_\alpha\}$ be an indexed family of elements in $\bigcap \mathcal{T}_\alpha$. Then each element of $\{U_\alpha\}$ is in each $\mathcal{T}_\alpha$ so arbitray unions and finite intersections of them must also be in each $\mathcal{T}_\alpha$ and thus in $\bigcap \mathcal{T}_\alpha$ so $\bigcap \mathcal{T}_\alpha$ is a topology on X.

For a simple counterexample to $\bigcup \mathcal{T}_\alpha$ being a topolgy, consider $X = \{a, b, c\}$, $\mathcal{T}_a = \{X, \emptyset, \{a\}\}$, and $\mathcal{T}_b = \{X, \emptyset, \{b\}\}$. $\mathcal{T}_a \bigcup \mathcal{T}_b$ contains $\{a\}$ and $\{b\}$ but not their union.

(b) $\bigcup \mathcal{T}_\alpha$ contains X so it forms a subbasis for X. Let$\mathcal{T}_3$ be the topology generated by this subbasis. $\mathcal{T}_3$ contains $\bigcup \mathcal{T}_\alpha$. Any topology containing $\bigcup \mathcal{T}_\alpha$ must also contain all unions of finite intersections of those sets so it must contain $\mathcal{T}_3$ so $\mathcal{T}_3$ must be the smallest topology containing $\bigcup \mathcal{T}_\alpha$. $\bigcap \mathcal{T}_\alpha$ is the largest thing contained in all $\mathcal{T}_\alpha$ by definition and it is a topology by part (a).

(c) We follow part (b): $\mathcal{T}_1 \bigcup \mathcal{T}_2 = \{X, \emptyset, \{a, b\}, \{b, c\}, \{a\}\}$ is a subbasis and the the topology generated by it, $\{X, \emptyset, \{a, b\}, \{b, c\}, \{a\}, \{b\}\}$, is smallest topology containing the union. $\mathcal{T}_1 \bigcap \mathcal{T}_2 = \{X, \emptyset, \{a\}\}$ is the largest topology contained in the intersection.

page 83 #7

Let $\mathfrak{B}_i$ be the basis given in the text for $\mathcal{T}_i$. We will make some use of Lemma 13.3.

$\mathfrak{B}_1 \subset \mathfrak{B}_2$ so $\mathcal{T}_1 \subset \mathcal{T}_2$, but not the reverse since $0 \in (-1,1) - K \in \mathfrak{B}_2$ but any element of $\mathfrak{B}_1$ containing 0 must also contain $1/n$ for some positive integer n. Let U be a nontrivial element of $\mathcal{T}_3$. Its complement contains a finite set of points so they are ordered. Thus one can express it as a union of open intervals (some of them may be semi-infinite but these too can be expressed as a countable union of open intervals) so $\mathcal{T}_3 \subset \mathcal{T}_1$. If $x \in B \in \mathfrak{B}_2$, B is either of the form (a,b) or $(a,b) - K$. In either case, we can choose c to be the larger of x or the greatest number $1/n$ less than x (here $n \in \mathbb{N}$). Then $x \in (c, x] \in \mathfrak{B}_4$ so $\mathcal{T}_2 \subset \mathcal{T}_4$. However, $3 \in (2,3] \in \mathfrak{B}_4$, but anythng in $\mathfrak{B}_2$ containing 3 must also contain something greater so $\mathcal{T}_4$ is not contained in $\mathcal{T}_2$. $1 \in \mathbb{R} - 0 \in \mathfrak{B}_3$ but any element of $\mathfrak{B}_5$ containing 1 must also contain 0 so $\mathcal{T}_3$ is not contained in $\mathcal{T}_5$. $(-\infty, 0) \in \mathcal{T}_5$ is nonempty but has an infinite complement so it's not in $\mathcal{T}_3$. Lastly, choose $y \in (-\infty, d) \in \mathfrak{B}_5$. Then $y \in (y-1, d) \in \mathfrak{B}_1$ so $\mathcal{T}_5 \subset \mathcal{T}_1$.

Combining all of this information we see that $\mathcal{T}_4$ contains all 4 other topologies, $\mathcal{T}_2$ contains all but $\mathcal{T}_4$, and $\mathcal{T}_1$ contains only $\mathcal{T}_3$ and $\mathcal{T}_5$ which don't contain any of the other topologies (since they are both contained in all of the other topologies but not each other).

page 83 #8

(a) Choose $x \in (a,b)$ where a and b are real. There exist rational numbers c and d such that $a < c < x < d < b$ so $x \in (c,d) \in \mathfrak{B}$ so this "new" topology contains the standard one. The reverse containment comes from the fact that $\mathfrak{B}$ is contained in the standard basis.

(b) $\pi \in [\pi, 4)$ which is in the basis for the lower limit topology, but if q and r are rational, $[q, r)$ either isn't contained in $[\pi, 4)$ (if $q < \pi$) or it doesn't contain π (if $q > \pi$).