The problems below are from old SB comps except otherwise stated.

(1) A topological group is a set G equipped with both a topology and a group structure, supposed compatible in the sense that the group operations $.: G \times G \longrightarrow G$, $(a, b) \mapsto a.b$ and $\cdot^{-1}: G \longrightarrow G$, $a \mapsto a^{-1}$ are continuous functions. Prove that the fundamental group of any path connected topological group is abelian.

(2) (no SB comps) Compute the fundamental group of $SL(2, \mathbb{R})$.

(3) Let X be the subspace of $\mathbb{R}^3$ which is a union of the unit sphere and five parallel lines each of which intersects the sphere in two distinct points. Compute the fundamental group of X.

(4) Let X denote 3 copies of the unit circle $\mathbb{S}^1$ identified along a common point.
  (a) Compute the fundamental group of X
  (b) Show that any continuous map $f: \mathbb{R}P^2 \longrightarrow X$ from the real projective plane is null homotopic.

(5) Let $C \subset \mathbb{R}^3$ be the unit circle in the xy-plane. Let $L \subset \mathbb{R}^3$ be the z-axis. Let $\mathbb{R}^3 \setminus (C \cup L)$ be the complement of these two curves. Compute the fundamental group of Y.

(6) (not from SB comps) Let U be an open set of $\mathbb{R}^3$. Let $C \subset U$ be a subset such that the subspace topology is discrete. Prove that $\pi_1(U)$ is isomorphic to $\pi_1(U \setminus C)$.

(7) Let A denote the subset of 3-space $\mathbb{R}^3$ consisting of the union of the z-axis, the unit circle centered at the origin in the x,y plane and the point $(3, 3, 0)$. Set $B = \mathbb{R}^3 \setminus A$. Show that $\pi_1(B)$ has a subgroup which is isomorphic to $\mathbb{Z}$.

(8) Let $A \subset \mathbb{R}^2$ be the compact planar region bounded by a regular hexagon. Let X be the surface with boundary obtained from A by identifying 2 pairs of opposite sides without creating Moebius bands. Compute the fundamental group of S.

(9) Let X denote the quotient space $\mathbb{R}^n / \sim$ where $\sim$ is the equivalence relation generated by $x \sim -x$ for all $x \in \mathbb{R}^n$. Prove that X is not a topological n-manifold if $n \geq 3$.

(10) Find the fundamental group of the complex polynomials $az^2 + bz + c$ of degree two with distinct roots.

(11) Let X and Y be Hausdorff topological space, and let $X \times Y$ be given the product topology. Let $f: X \longrightarrow Y$ be a function and let $\Gamma_f \subset X \times Y$ be its graph.
  (a) If f is continuous, show that Γ_f is closed.
  (b) If X and Y are compact show that the converse of a) is true.
  (c) Does de converse of (a) remain true if we merely assume that X is compact?

(12) (not SB comps) Find the fundamental group of $O(3)$ the group of orthogonal 3×3 matrices with coefficients in $\mathbb{R}$. (First find the fundamental group of $SO(3)$

The problems below are for other universities comps or other sources.

(1) Let $G = SL(2,\mathbb{R})$ be the topological group of 2×2 matrices with determinant one. Consider the subrgoup $H = SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ of all matrices with integer entries. Prove that the quotient space G/H with the quotient-topology is normal, locally compact but not compact.
(2) Let $\mathbb{S}^n$ denote the sphere and suppose that $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is a continuous map without fixed points. Prove that f is homotopic to the antipodal map $f(x) = -x$.
(3) Prove that the space of continuous maps from the real line to the unit interval is separable in the compact-open topology (uniform convergence on compact sets) and is not separable in the uniform topology.
(4) Prove that the open cylinder with one point removed and the torus T^2 with one point removed are homotopically equivalent and calculate the fundamental group of those spaces.
(5) Prove that the projective plane is not contractible and is not homotopically equivalent to a sphere or a torus of any dimension.
(6) Is the fundamental group of the Hawain ring finite? free? countable or uncountable? Is the Hawaing ring homeomorphic to a bouquet of infinite circles? Is it homeomorphic to the one point compactification of $\mathbb{R} \setminus \mathbb{Z}$?
(7) Find all 2-fold coverings of the figure eight.
(8) Compute the fundamental group of the manifold obtained from $T^2 \times I$ by identifying the opposite faces by the glueing map $(1,0) \longrightarrow (2,1)$ and $(0,1) \longrightarrow (1,1)$.
(9) Show that any two embeddings of a connected closed set X in the two sphere have homeomorphic complements.
(10) Show that $\mathbb{C}P^2$ does not cover any manifold other than itself.
(11) Let E be the total space of a covering space of the Klein bottle. Classify E up to homotopy equivalence.
(12) Denote by G the fundamenatl group of the figure eight. Describe the covering space of the figure eight corresponding to the subgroup $[G,G]$.
(13) Hatcher Problem 6 and 21, section 1.3