MAT 364 Extra homework 1

Functions and relations

October 11, 2012

A *relation R on a set X* is a subset of ordered pairs of elements of X, that is $R \subset XxX$.

A relation R is *reflexive* if (x, x) belongs to R for all x in X.

An example of a relation that is not reflexive is the following: X is a set of people, R_1 is the set of pairs (x, y) of people in X such that x is a brother or sister of y.

A relation R is *symmetric* if and only if the following holds: If a pair (x, y) belongs to R then the pair (y, x) belongs to R.

The relation R_1 is symmetric.

A relation R is *transitive* if and only if the following holds: If the pairs (x, y) and (y, z) belong to R then the pair (x, z) belongs to R.

An example of a relation that is reflexive, symmetry and transitive is the following: $X = \mathbb{Z}$, $(x, y) \in R_2$ if x and y have the same parity.

Recall that a relation R is called an *equivalence relation* if and only if it is symmetric, reflexive and transitive.

If R is an equivalence relation on a set A, and a is an element of A, the equivalence class of a is the set $\{b \in A \text{ such that } (a, b) \in R\}$

A *partition of a set A* is a disjoint collection of nonempty subsets of A whose union is the whole A. In other words, it is al ist of subsets X_α of X, for α is some set L of labels so that for every x in X there is one and only one label α so that x belongs to X_α.

Consequence: There is a function p from X to L, namely, $p(x) = \alpha$, where $x \in X_\alpha$.

Workhouse philosophy: In Mathematics, wide use is made of the idea to try to take interesting properties or structures on X and induce them on L using p.

A *function* f from a set X to a set Y, is a correspondence that assigns to each element x of X, a unique element of $f(x)$ of Y.

We can think of a function from a space X to itself, as a relation R which satisfies that for all x in X, there is one and only one $y \in X$ such that $(x, y) \in R$.

(1) Give an example of a relation on X that is reflexive and an example of a relation that it is not reflexive, in each of the following cases:

   (a) X is a set of people.

   (b) X is $\mathbb{R}$, the set of real numbers.

(2) Give an example of a relation on X that is not symmetric and an example of a relation that it is symmetric, in each of the following cases:

   (a) X is a set of people.

   (b) X is $\mathbb{R}$, the set of real numbers.

(3) Give an example of a relation that is transitive and an example of a relation that it is not transitive.

(4) Give an example of a relation that is symmetric and transitive, but not reflexive.

(5) Give an example of a relation that is reflexive but not transitive.

(6) Give three examples of equivalence relations.

(7) Show that an equivalence relation determines a partition.

(8) Show that a partition determines an equivalence relation.

(9) Show the partitions determined by the equivalence relations you found in problem 6 above.

(10) Define two points on $\mathbb{R}^2$ to be equivalent if they have the same x coordinate. Is it an equivalence relation? If the answer is yes, find the partition it determines.

(11) Consider the partition of $\mathbb{R}^2$ defined by all lines parallel to the line $y = x$. Describe (in terms of the coordinates of the points) the equivalence relation this partition determines.

(12) Define two points (p, q) and (r, s) of the plane to be equivalence if $p - q^3 = r - s^2$. Check that this is an equivalence relation and describe the equivalence classes.

(13) Short that if a a function is a symmetric relation then it must be bijective.

(14) Show that if a function F is a symmetric and transitive relation then it must be an involution. (An involution is a function that satisfies $F \circ F = Identity$.

(15) Show that if a function is reflexive then it must be the identity mapping.

(16) A function between two finite sets can be described by the enumerating the set of pairs, or by a diagram. Set $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$. In the following cases, if possible, give examples (in both forms, the diagrama and enumeration) . If it is not possible explain why.

   (a) An injective function from A to B.
   (b) An injective function from B to A.
   (c) An injective function from A to A.
   (d) A surjective function from B to A.

(17) Consider a surjective function f from A to B. Define an equivalence relation R on A by setting $(a, a') \in R$ if and only if $f(a) = f(a')$. Show that R is an equivalence relation, and that there is a bijective correspondence between B and the set $A*$ of equivalence classes. of R.