79.58 Prime Values of Polynomials

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$$s(s-a) = p(p+q) \times q(p-q) = pq(p^2-q^2). \tag{4}$$

The usual formula to find the area of a triangle is *Area* = ½ *base* × *height*. Using this and (4) we have

$$A = \tfrac{1}{2}(p^2-q^2) \times 2pq = pq(p^2-q^2) = s(s-a).$$

I later realised that there was no need to use Pythagorean triples, where all the sides are integers, only Pythagoras' Theorem. In fact formula (1) works if and only if the triangle is right-angled. If (1) holds then using Heron's formula for the area of a triangle with sides of lengths a, b and c,

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

we have

$$\sqrt{s(s-a)(s-b)(s-c)} = s(s-a).$$

Squaring both sides and dividing by $s(s-a)$ we obtain

$$(s-b)(s-c) = s(s-a).$$

Expanding the brackets and rearranging gives

$$bc = s(b+c-a)$$

where $2s = a+b+c$. Thus

$$2bc = (b+c+a)(b+c-a)$$

which simplifies to

$$a^2 = b^2 + c^2.$$

By the converse of Pythagoras' Theorem the triangle is right-angled.

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79.58 Prime values of polynomials

The thoughts which follow were triggered by reading the discussion of a computer based approach to the factorisation of quadratics in the excellent 1992 Mathematical Association report *Computers in the mathematics classroom*. In this, on pages 68 and 69, the values of x^2+5x+4 and x^2+5x+5 for $x = 0, 1, \dots, 9$ are contrasted, it being noted that in the latter case 'the predominance of prime number [values] reflects the impossibility of expressing x^2+5x+5 as the product of linear factors with integral coefficients'.

In what follows, we shall suppose all polynomials to have integer coefficients and we recall Gauss' Lemma – that factorisations of such polynomials over $\mathbb{Q}$ and $\mathbb{Z}$ are equivalent.

Result 1 If p is a quadratic and $p(x)$ is prime for 5 integer values of x, then p is irreducible.

For if $p(x) = a(x)b(x)$ with a, b linear, then $a(x) = \pm 1$, $b(x) = \pm 1$ are satisfied by at most 4 distinct integer values of x: and if $p(x)$ is prime for some x then necessarily $a(x) = \pm 1$ or/and $b(x) = \pm 1$. This result – and its proof – generalises immediately to give:

Result 2 If p is a polynomial of degree n and $p(x)$ is prime for $2n + 1$ integer values of x, then p is irreducible.

(Just observe that $a(x) = \pm 1, b(x) = \pm 1$ have at most $2(\deg(a) + \deg(b)) = 2n$ integer solutions.)

These conditions are, of course, far from necessary for irreducibility: for example, the irreducible monic quadratic $x(x + 1) + 4$ takes just (non-prime) even values for integer x and, for prime $p \geqslant 5$, the monic polynomial $x^p - (p + 1)x + 6p$ (which is irreducible by Eisenstein's criterion with prime 2) takes values which, by Fermat's little theorem, are all composite and multiples of p.

We conclude with two open (hard?) problems:

Problem 1 For each n, does there exist a reducible polynomial p of degree n for which $p(x)$ is prime for $2n$ values of x? (In this sense, Result 2 would then be best possible. To get you started, $(x - 1)(x - 5)$ will do for $n = 2$.)

Problem 2 With extra hypotheses on p, more can be asserted: thus if a quadratic with positive coefficients is prime for 3 positive integer values, then it is irreducible by a similar argument to that given for Result 1. What must 3 be replaced by in order to guarantee that a polynomial of degree n with positive coefficients is irreducible?

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79.59 Balancing and golden rectangles

It is always intriguing to meet old friends in new contexts. Consider the following standard-looking problem: What $x \times x$ square must be removed from the 1×1 square lamina shown in Figure 1 so that the remaining gnomon has centre of mass at the corner of the square removed?

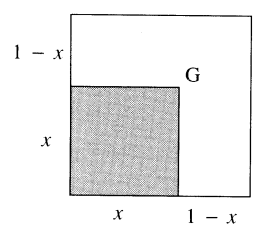

FIGURE 1

Taking moments about the left-hand edge in the usual way, this requires that:

$$\underset{\text{removed square}}{\tfrac{1}{2}x \times x^2} + \underset{\text{gnomon}}{x(1 - x^2)} = \underset{\text{original square}}{\tfrac{1}{2} \times 1}$$

whence $x^3 - 2x + 1 = 0$ or $(x - 1)(x^2 + x - 1) = 0$.

Thus either $x = 1$ (interpret) or $x = \frac{1}{2}(\sqrt{5} - 1) = 1/\tau$ where τ, the