The Magic Hexagram
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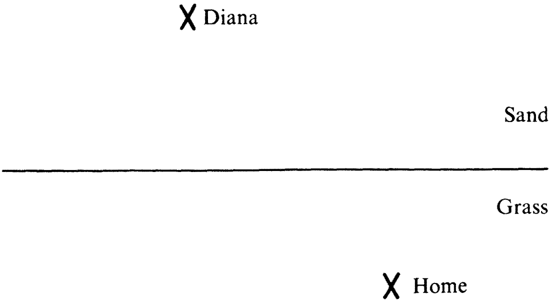
We now let θ tend to α and so P tends to R and in the limit PRQ becomes the tangent to the circle at R. In the limiting position we can calculate OQ and it is $1/\sin\alpha$. but OQ is the speed of Diana on the road and so

$$\lim_{\theta\to\alpha^-}\frac{\tan\alpha-\tan\theta}{1/\cos\alpha-1/\cos\theta}=\frac{1}{\sin\alpha}$$

An Open Problem

Is it possible to apply the theorem to find Diana's quickest way home in the following situation, where Diana runs at 7 mph on grass and 5 mph on sand?



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The magic hexagram

BRIAN BOLT, ROGER EGGLETON AND JOE GILKS

It was at the Fifth Southeast Asian Conference on Mathematical Education (June 1990, in Brunei Darussalam) that Brian Bolt posed the question: *Can the numbers 1 to 12 be arranged within the hexagram below (Figure 1) to give equal sums in the six indicated directions?* This is the *Magic Hexagram Problem.*

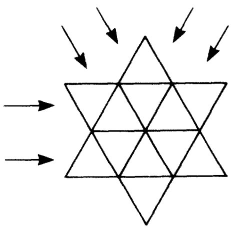


Figure 1

During and immediately after the Conference the three authors got to work on the problem, and now (we believe for the first time) the answer is known.

There are 12! ways of allocating the numbers to the triangles of the hexagram, but rotation and reflection symmetries reduce this to $11! = 39{,}916{,}800$ cases to consider. However ways can be found to restrict the search.

Denote the six perimeter entries by $p_1, p_2 \ldots p_6$ and the remaining inner entries by $i_1, i_2 \ldots i_6$. It is clear that

$$2 \sum p_j + 3 \sum i_j = 6S$$

where S is the common sum. But summing the numbers from 1 to 12 yields

$$2(\sum p_j + \sum i_j) = 156$$

so

$$\sum i_j = 6(S - 26).$$

From this we find

(1) $\sum i_j$ (and hence $\sum p_j$) is divisible by 6;

(2) $30 \leqslant S \leqslant 35$ (because $24 \leqslant \sum i_j \leqslant 54$).

Next observe that if a solution can be found, replacing each entry x_j by $13 - x_j$ permutes the entries and yields a second solution. This interchanges odd and even numbers and gives a solution which is distinct from any of the rotations or reflections of the original solution, because it gives a new common sum equal to $65 - S \neq S$. (We shall call these two potential solutions *complementary*.) Therefore the total number of distinct solutions (without counting rotations and reflections as distinct) must be even. But this doesn't yet prove that there is a solution—zero is an even number!

The problem reduces to one of manageable size when it is noted that the distribution of odd and even numbers must be consistent with the required sums being equal. After rejecting equivalents with respect to rotation, reflection and complementation (the odd/even interchange) only three possibilities can arise. They are shown in Figure 2, where each 0 corresponds to an even number and each 1 corresponds to an odd.

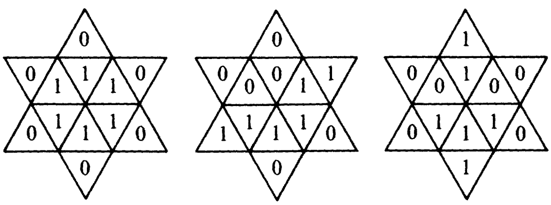
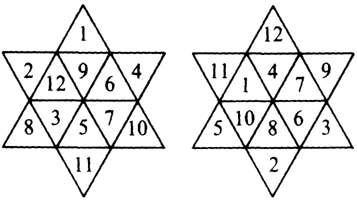


Figure 2 a, b, c.

Alternatively, the configurations in Figure 2 may be interpreted as Magic Hexagram solutions in modulo 2 arithmetic. (Similarly, putting 0 on each black square and 1 on each white square of a chessboard gives an 8×8 Magic Square solution in modulo 2 arithmetic.)

The Magic Hexagram problem is now reduced to manageable proportions, and a modest computer search based on the three configurations in Figure 2 reveals exactly one solution, corresponding to Figure 2(c); together with its complementary solution, this gives the only possible solutions. (See Figure 3.) When we discovered the beautiful configurations in figure 3, we rushed from our baths into the streets shouting "Eureka!"



(a) $S = 33$ (b) $S = 32$

Figure 3

We have also completed a computer-independent solution of the Magic Hexagram problem. Our approach to this study considered modulo 3, modulo 4 and modulo 6 solutions. Classroom activities involving these are suggested and recommended. For example, it is not difficult to derive such solutions from Figure 3, but there are others as well. Students should note how adding one (in the appropriate modular arithmetic) to each entry gives another solution.

Allowing modular arithmetic solutions opens a Pandora's box. Applying this idea to magic squares, cubes and so on gives a fertile area for investigations of the type that students of all ages will surely enjoy.

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MA culpa

"$e + 1 = 0$." From the MA's "Publication, Teaching Aids and Accessories", sent in by Humphrey Oates who is also responsible for the title of this gleaning.