SIMPLE ANALYTIC PROOF OF THE PRIME NUMBER THEOREM

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The magnificent prime number theorem has received much attention and many proofs throughout the past century. If we ignore the (beautiful) elementary proofs of Erdős [1] and Selberg [6] and focus on the analytical ones, we find that they all have some drawback. The original proofs [7] of Hadamard and de la Vallée Poussin were based, to be sure, on the nonvanishing of $\zeta(z)$ in Re $z \geqslant 1$, but they also required annoying estimates of $\zeta(z)$ at ∞, the reason being that formulas for coefficients of Dirichlet series involve integrals over *infinite* contours (unlike the situation for power series) and so effective evaluation requires estimates at ∞.

The more modern proofs, due to Wiener [2] and Ikehara [8] (see also Heins's book [3]) do get around the necessity of estimating at ∞ and are indeed based only on the appropriate nonvanishing of $\zeta(z)$, but they are tied to certain results on Fourier transforms.

We propose to return to contour integral methods so as to avoid Fourier analysis, and also to use finite contours so as to avoid estimates at ∞. Of course certain errors are introduced thereby, but the point is that these can be effectively estimated away by elementary arguments.

So let us begin with the well-known fact [7] about the ζ-function:

$$(z-1)\zeta(z) \text{ is analytic and zero free throughout } \operatorname{Re} z \geqslant 1. \tag{1}$$

This will be assumed throughout and will allow us to give our proof of the prime number theorem.

In fact we give two proofs. The first one is the shorter and simpler of the two, but we pay a price in that we obtain one of Landau's equivalent forms of the theorem rather than the standard form, $\pi(N) \sim N/\log N$. Our second proof is a more direct assault on $\pi(N)$ but is somewhat more intricate than the first. Here we find some of Tchebychev's elementary ideas very useful.

Basically our novelty consists in using a modified contour integral,

$$\int_\Gamma f(z)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz,$$

rather than the classical one, $\int_C f(z)N^z z^{-1}dz$. The method is rather flexible, and we could use it to directly obtain $\pi(N)$ by choosing $f(z)=\log \zeta(z)$. We prefer, however, to derive both proofs from the following convergence theorem. Actually, this theorem dates back to Ingham [9], but his proof is à la Fourier analysis and is much more complicated than the contour integral method we now give.

THEOREM. *Suppose* $|a_n| \leqslant 1$ *and form the series* $\sum a_n n^{-z}$ *which clearly converges to an analytic function* $F(z)$ *for* Re $z>1$. *If, in fact,* $F(z)$ *is analytic throughout* Re $z \geqslant 1$, *then* $\sum a_n n^{-z}$ *converges throughout* Re $z \geqslant 1$.

Proof of the convergence theorem.. Fix a w in Re $w \geqslant 1$. Thus $F(z+w)$ is analytic in Re $z \geqslant 0$. We choose an $R \geqslant 1$ and determine $\delta=\delta(R)>0$, $\delta \leqslant \frac{1}{2}$ and an $M=M(R)$ so that

$$F(z+w) \text{ is analytic and bounded by } M \text{ in } -\delta \leqslant \operatorname{Re} z, |z| \leqslant R. \tag{2}$$

Now form the counterclockwise contour Γ, bounded by the arc $|z|=R$, Re $z>-\delta$, and the

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segment $\operatorname{Re} z = -\delta, |z| \leqslant R$. Also denote by A and B, respectively, the parts of Γ in the right and left half-planes.

By the residue theorem we have

$$2\pi i F(w) = \int_\Gamma F(z+w) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz. \tag{3}$$

Now on A, $F(z+w)$ is equal to its series, and we split this into its partial sum $S_N(z+w)$ and remainder $r_N(z+w)$. Again by the residue theorem we have

$$\int_A S_N(z+w) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz = 2\pi i S_N(w) - \int_{-A} S_N(z+w) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz,$$

with $-A$ denoting as usual the reflection of A through the origin. Thus, changing z into $-z$, this can be written as

$$\int_A S_N(z+w) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz = 2\pi i S_N(w) - \int_A S_N(w-z) N^{-z} \left(\frac{1}{z} + \frac{z}{R^2} \right) dz. \tag{4}$$

Combining (3) and (4) gives

$$\begin{aligned} 2\pi i (F(w) - S_N(w)) = & \int_A \left(r_N(z+w) N^z - \frac{S_N(w-z)}{N^z} \right) \left(\frac{1}{z} + \frac{z}{R^2} \right) dz \\ & + \int_B F(z+w) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz, \end{aligned} \tag{5}$$

and to estimate these integrals we record the following (here as usual we write $\operatorname{Re} z = x$, and we use the notation $\alpha \ll \beta$ to mean simply that $|\alpha| \leqslant |\beta|$):

$$\frac{1}{z} + \frac{z}{R^2} = \frac{2x}{R^2} \text{ along } |z| = R \text{ (in particular on } A), \tag{6}$$

$$\frac{1}{z} + \frac{z}{R^2} \ll \frac{1}{\delta} \left(1 + \frac{|z|^2}{R^2} \right) \leqslant \frac{2}{\delta} \text{ on the line } \operatorname{Re} z = -\delta, |z| \leqslant R, \tag{7}$$

$$r_N(z+w) \ll \sum_{n=N+1}^{\infty} \frac{1}{n^{x+1}} \leqslant \int_N^\infty \frac{dn}{n^{x+1}} = \frac{1}{xN^x}, \tag{8}$$

$$S_N(w-z) \ll \sum_{n=1}^{N} n^{x+1} \leqslant N^{x-1} + \int_0^N n^{x-1} dn = N^x \left(\frac{1}{N} + \frac{1}{x} \right). \tag{9}$$

By (6), (8), (9) we have, on A,

$$\left(r_N(z+w) N^z - \frac{S_N(w-z)}{N^z} \right) \left(\frac{1}{z} + \frac{z}{R^2} \right) \ll \left(\frac{1}{x} + \frac{1}{x} + \frac{1}{N} \right) \frac{2x}{R^2} \leqslant \frac{4}{R^2} + \frac{2}{RN},$$

and so by the "maximum times length" estimate (M-L formula) for integrals we obtain

$$\int_A \left(r_N(z+w) N^z - \frac{S_N(w-z)}{N^z} \right) \left(\frac{1}{z} + \frac{z}{R^2} \right) dz \ll \frac{4\pi}{R} + \frac{2\pi}{N}: \tag{10}$$

Next by (2), (6), and (7) we obtain

$$\begin{aligned} \int_B F(z+w) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz & \ll \int_{-R}^{R} M \cdot N^{-\delta} \cdot \frac{2}{\delta} dy + 2M \int_{-\delta}^{0} N^x \frac{2|x|}{R^2} \frac{3}{2} dx \\ & \leqslant \frac{4MR}{\delta N^\delta} + \frac{6M}{R^2 \log^2 N}. \end{aligned} \tag{11}$$

Inserting the estimates (10) and (11) into (5) gives

$$F(w)-S_N(w) \ll \frac{2}{R}+\frac{1}{N}+\frac{MR}{\delta N^{\delta}}+\frac{M}{R^2\log^2 N}$$

and if we fix $R=3/\epsilon$ we note that this right-hand side is $<\epsilon$ for all large N. We have verified the very definition of convergence!

First Proof of the Prime Number Theorem. Landau has pointed out that the convergence of $\sum \mu(n)/n$ is equivalent to the prime number theorem. Since $\sum \mu(n)/n^z = 1/\zeta(z)$ for Re $z>1$, however, (1) ensures that the hypotheses of our theorem hold, and Landau's form of the prime number theorem follows immediately.

Second Proof of the Prime Number Theorem. In this section we begin with Tchebychev's observation [5] that

$$\sum_{p<n} \frac{\log p}{p} - \log n \quad \text{is bounded,} \tag{12}$$

which he derives in a direct elementary way from the prime factorization of $n!$.

The point is that the prime number theorem is easily derived from

$$\sum_{p<n} \frac{\log p}{p} - \log n \quad \text{converges to a limit,} \tag{13}$$

by a simple summation by parts, which we leave to the reader. Nevertheless the transition from (12) to (13) is not a simple one and we turn to this now.

So form, for Re $z>1$, the function

$$f(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}\left(\sum_{p<n}\frac{\log p}{p}\right)=\sum_p \frac{\log p}{p}\left(\sum_{n>p}\frac{1}{n^z}\right).$$

Now

$$\sum_{n>p}\frac{1}{n^z}=\frac{1}{(z-1)p^{z-1}}+z\int_p^{\infty}\frac{1-\{t\}}{t^{z+1}}\,dt=\frac{p}{(z-1)}\left(\frac{1}{p^z-1}+A_p(z)\right)$$

where $A_p(z)$ is analytic for Re$z>0$ and is bounded by

$$\frac{1}{p^x(p^x-1)}+\frac{|z(z-1)|}{xp^{x+1}}.$$

Hence

$$f(z)=\frac{1}{z-1}\left(\sum_p \frac{\log p}{p^z-1}+A(z)\right),$$

where $A(z)$ is analytic for Re $z>\frac{1}{2}$ by the Weierstrass M-test.

By Euler's factorization formula, however, we recognize that

$$\sum_p \frac{\log p}{p^z-1}=\frac{-d}{dz}\log\zeta(z); \tag{14}$$

and so we deduce, by (1), that $f(z)$ is analytic in Re $z \leqslant 1$ except for a double pole with principal part $1/(z-1)^2+c/(z-1)$, at $z=1$. Thus if we set

$$F(z)=f(z)+\zeta'(z)-c\zeta(z)=\sum\frac{a_n}{n^z}, \qquad \text{where } a_n=\sum_{p<n}(\log p)/p-\log n-c,$$

we deduce that $F(z)$ is analytic in Re $z \geqslant 1$.

From (12) and our convergence theorem, then, we conclude that

$$\sum \frac{a_n}{n} \text{ converges,}$$

and from this and the fact, from (14), that $a_n + \log n$ is nondecreasing we proceed to prove $a_n \to 0$.

By applying the Cauchy criterion we find that, for N large, we have both

$$\sum_{N}^{N(1+\epsilon)} \frac{a_n}{n} \leqslant \epsilon^2, \tag{15}$$

and

$$\sum_{N(1-\epsilon)}^{N} \frac{a_n}{n} \geqslant -\epsilon^2. \tag{16}$$

In the range N to $N(1+\epsilon)$ we have, by (14), that $a_n \geqslant a_N + \log(N/n) \geqslant a_N - \epsilon$ and so $\Sigma_N^{N(1+\epsilon)} a_n/n \geqslant (a_N - \epsilon)\Sigma_N^{N(1+\epsilon)} 1/n$ and (15) yields

$$a_N \leqslant \epsilon + \frac{\epsilon^2}{\sum\limits_{N}^{N(1+\epsilon)} \frac{1}{n}} \leqslant \epsilon + \frac{\epsilon^2}{N\epsilon/N(1+\epsilon)} = 2\epsilon + \epsilon^2. \tag{17}$$

Similarly in $[N(1-\epsilon), N]$ we have $a_n \leqslant a_N + \log(N/n) \leqslant a_N + \epsilon/(1-\epsilon)$ so that

$$\sum_{N(1-\epsilon)}^{N} \frac{a_n}{n} \leqslant \left(a_N + \frac{\epsilon}{1-\epsilon}\right) \sum_{N(1-\epsilon)}^{N} \frac{1}{n}$$

and (16) gives

$$a_N \geqslant \frac{-\epsilon}{1-\epsilon} - \frac{\epsilon^2}{\sum\limits_{N(1-\epsilon)}^{N} \frac{1}{n}} \geqslant -\frac{\epsilon}{1-\epsilon} - \frac{\epsilon^2}{N\epsilon/N} = \frac{\epsilon^2 - 2\epsilon}{1-\epsilon}. \tag{18}$$

Taken together (17) and (18) establish that $a_N \to 0$ and so (13) is proved.

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MISCELLANEA

43. If you ask mathematicians what they do, you always get the same answer; they think. They are trying to solve difficult and novel problems. (They never think about ordinary problems—they just write down the answers.)

—M. Evgrafov, *Literaturnaya Gazeta*, no. 49 (1979) 12.