One can, however, exploit L'Hôpital's rule to motivate a circle of ideas which lead to an important result rarely taught in elementary calculus, namely the intermediate value theorem for derivatives. The procedure is as follows:

Let f be differentiable on an interval containing $x = a$. Then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

and (since f is continuous at $x = a$) both the numerator and the denominator of the difference quotient approach zero. L'Hôpital's rule then gives

$$f'(a) = \lim_{x \to a} f'(x), \tag{$*$}$$

a result which apparently shows that f' is continuous at $x = a$.

At this stage it is natural to present $f(x) = x^2\sin(1/x)$ with $f(0) = 0$ in order to show that not every differentiable function has a continuous derivative. One may then challenge students to find the flaw in the argument or to explain why, in view of the example, that equation $(*)$ is valid. A properly conducted discussion should lead students to understand that, because of the hypothesis of L'Hôpital's rule, equation $(*)$ is true only in the sense that "if $\lim_{x\to a} f'(x)$ exists, then $\lim_{x\to a} f'(x) = f'(a)$."

The existence of $\lim_{x\to a} f'(x)$ for each a in an interval I means that f' has the intermediate value property (i.e., f' assumes all values between any two values it takes on I). Actually, f' cannot fail to have this property, since the intermediate value theorem for derivatives asserts that, if f' exists everywhere on I and takes on any two values on I, then f' takes every possible value between them. [See, for example, W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill (1964), p. 93.]

The Sum of the Reciprocals of the Primes

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For a long time it has been known that the sum of the reciprocals of the primes $\sum 1/p = 1/2 + 1/3 + 1/5 + 1/7 + 1/11 + \cdots$ forms a divergent series. Recently a very neat, new proof has been devised by Frank Gilfeather and Gary Meisters, who are to be thanked for their permission to present it here.

For a given integer $n \geqslant 2$, we take the set of all primes $p \leqslant n$ and consider the product

$$\prod_{p \leqslant n} \left(\frac{p}{p-1} \right) = \prod_{p \leqslant n} \left(\frac{1}{1 - 1/p} \right) = \prod_{p \leqslant n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right). \tag{1}$$

Since each $k \leqslant n$ is a product of various powers of certain primes less than or equal to n, we are certain to have $1/k$ as one of the terms in the above multiplication

$$\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right)\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \right) \cdots .$$

This is true for all $k \leqslant n$. Therefore, it follows from (1) that

$$\prod_{p \leqslant n} \left(\frac{p}{p-1} \right) > \sum_{k=1}^{n} \frac{1}{k}.$$

Since the logarithm function is monotonic (preserves inequalities), this gives

$$\sum_{p \leqslant n} (\log p - \log(p-1)) > \log\left(\sum_{k=1}^{n} \frac{1}{k} \right). \tag{2}$$

On the other hand,

$$\sum_{p \leqslant n} (\log p - \log(p-1)) = \sum_{p \leqslant n} \left(\int_{p-1}^{p} \frac{1}{x}\, dx \right) < \sum_{p \leqslant n} \left(\frac{1}{p-1} \right) \leqslant \sum_{p \leqslant n} \frac{2}{p}. \tag{3}$$

Combining inequalities (2) and (3), we arrive at

$$\sum_{p \leqslant n} \frac{1}{p} > \frac{1}{2} \log\left(\sum_{k=1}^{n} \frac{1}{k} \right). \tag{4}$$

The right-hand side of (4) increases without limit as n increases. Therefore $\sum 1/p$ diverges.

Remark. It may be surprising to note that the sum of the reciprocals of the twin primes (i.e., pairs of primes which differ by 2, such as 11, 13 or 17, 19) is a convergent series. [See, for example, E. Landau, *Elementary Number Theory*, 2nd ed., Chelsea (1966), pp. 94–103.]

"Why Can't We Trisect an Angle This Way?"

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Richard L. Francis [TYCMJ 9, 2 (1978) 75–80] refers to the usual nonconstructible root argument to show the impossibility of the Euclidean angle trisection. This same discussion still meets resistance in geometry classes. The question often raised is "What's wrong with trisecting the base BC of an isosceles triangle ABC at points D and E, and then claiming that segments AD and AE trisect the given angle BAC?" (Since $\triangle ABD \cong \triangle ACE$, we do have $\angle BAD = \angle EAC$.) The question is not all that unreasonable considering the validity of the analogous bisection of $\angle A$ through bisecting BC.

One form of analysis could involve trigonometry. For instance, let $\angle BAC = 60°$, and take $AB = AC = 3$ (so that $BD = DE = EC = 1$). Since $\angle ABD = 60°$, the law of cosines applied to $\triangle ABD$ yields $AD = \sqrt{7}$. Then the law of sines yields

$$\sin BAD = \frac{1}{2}\sqrt{\frac{3}{7}} \quad \text{and} \quad \angle BAD \approx 19.1°,$$