Magic Cubes and Hypercubes
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Source: *The Mathematical Gazette*, Vol. 58, No. 403 (Mar., 1974), pp. 25-27
Published by: The Mathematical Association
Stable URL: http://www.jstor.org/stable/3615474
Accessed: 24/03/2010 21:40

Magic cubes and hypercubes

S. N. COLLINGS

Because of 'modern mathematics' most people are aware of different number scales. Ordinarily there is the decimal scale, while computers use the binary system and sometimes, I believe, the octal system based on 8s. Other scales may appear to be freakish, or mere excuses for arithmetical manipulations; yet at times they can be useful.

Another of the current interests is congruence arithmetic. If the integers a and b have the same remainder when divided by m, we say that a and b are *congruent* modulo m. In symbols,

$$a \equiv b \pmod{m} \text{ if } a - b \text{ is a multiple of } m.$$

Everyone is aware that the days of the week represent a modular situation; that is that in a given month, date a is the same day of the week as date b if and only if

$$a \equiv b \pmod{7}.$$

However, this does not appear to be any improvement over colloquial English, and generally there must be the impression that congruence arithmetic (also called modular arithmetic) is one of those mathematical abstractions which from its artificiality cannot tell us anything about 'real arithmetic' and the real world. Once again you would be surprised. Below we shall combine the ideas of different bases and modular arithmetic to produce new results in an extension of magic squares.

Magic squares are familiar enough; every line of numbers (parallel to a side) adds up to the same sum. Less familiar are magic cubes possessing exactly the same property. The figure shows a magic cube of dimensions 3^3. In addition to the property stated, every line through the centre cell of this cube is also magic, whether it is parallel to a side or not.

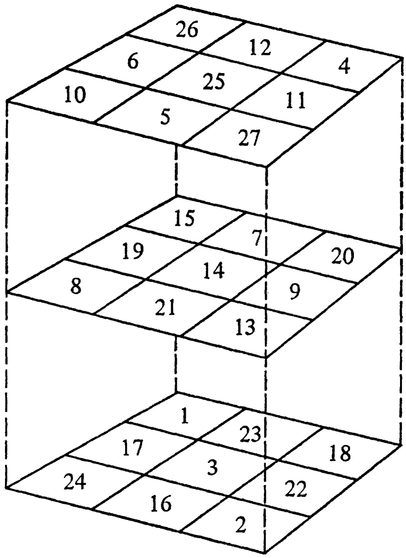
It is relatively easy to generate a magic hypercube with p^n cells in n dimensions, where p is an odd prime but not a factor of $(n+1)$. We work in the scale of p, and we add and multiply (mod p) without carrying from one column to the next. Thus

$$\begin{array}{r} 212 \\ +211 \\ \hline 120 \end{array} \qquad \begin{array}{r} 212 \\ \times\ 2 \\ \hline 121 \end{array}$$

Taking a corner cell as origin, any cell in the hypercube has coordinates of the form $(x_1, x_2, \ldots, x_n)$ where each x is a residue (mod p). We denote the number to go into this cell by $f(x_1, x_2, \ldots, x_n)$.

It remains to define the function f over the hypercube:

(i) For all $(x_1, x_2, \ldots, x_n)$, $f(x_1, x_2, \ldots, x_n)$ is an n-digit number in the scale of p.
(ii) $f(0, 0, \ldots, 0) = 00\ldots0$.
(iii) $f(1, 0, 0, \ldots, 0) = 211\ldots1$,
$f(x_1, 0, 0, \ldots, 0) = x_1 \times 211\ldots1$.
(iv) Similarly for $f(0, x_2, 0, \ldots, 0), f(0, 0, x_3, \ldots, 0), \ldots, f(0, 0, 0, \ldots, x_n)$.
(v) $f(x_1, x_2, \ldots, x_n) = f(x_1, 0, 0, \ldots, 0) + f(0, x_2, 0, \ldots, 0) + \ldots$
$+ f(0, 0, 0, \ldots, x_n)$.



If we do this with $p = 3$, $n = 3$, add 1 to all the cell numbers and then convert to the scale of 10, we get the magic cube already mentioned.

In conclusion, a word about the peculiar addition and multiplication which never "carry 1". These do not look like any operations we have seen before; the reason is simple—they are not really arithmetical operations at all. They are contracted representations of operations on polynomials.

Given any 3-digit number abc, it is obviously closely associated with the polynomial $ax^2 + bx + c$. Indeed, from another point of view, $ax^2 + bx + c$ is merely the written-out version of the number abc in the scale of x; but x remains a variable. Adding the numbers abc, $a'b'c'$ without carrying is equivalent to adding the corresponding polynomials, and it gives $(a + a')x^2 + (b + b')x + (c + c')$. Doubling the number is equivalent to doubling the polynomial, and it gives $(2a)x^2 + (2b)x + 2c$. Obviously with the polynomials there is no question of carrying, for no number of xs is algebraically equivalent to any x^2. Finally, if the coefficients a, b, c of the polynomials belong to a congruence field modulo 3, then we get just those results mentioned earlier on. Thus

$$(2x^2+x+2)+(2x^2+x+1)\equiv x^2+2x,$$

and

$$2(2x^2+x+2)\equiv x^2+2x+1.$$

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Function boxes: a model for differentiation

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The paper begins with a description of a diagrammatical model which I used when lecturing on elementary calculus to a class of engineers. The second part suggests how the use of the model might be extended to provide a framework on which one could build up the theory of differentiation of functions of one or more real variables. For the purposes of this article it is assumed that the reader is familiar with this theory—the paper, therefore, merely gives the outline of a possible approach. The keynote of the lecture course to the engineers was 'plausibility' rather than 'rigour' and my use of 'function boxes' was sparked off by the fact that the class were using flow diagrams and 'black boxes' in a number of other courses—most noticeably in that given by the person who lectured immediately before me and who, in his progress up the technological scale to the electronic computer, had clearly not encountered that mundane educational aid, the blackboard duster. The grand finale of the one-term course was to be an introduction to partial differentiation and, in particular, the chain rule—a piece of mathematics which students often find difficult and which I hoped to make easier by means of an approach using a simple type of 'flow' diagram.

The idea of a function box is to be found in SMP Book 2, where a function is seen as a kind of sausage machine—into the machine called 'square' goes 5, and out comes 25 (Fig. 1). Alternatively, in place of 'square' one can write down the appropriate formula and can generalise the situation by allowing an arbitrary input (Fig. 2).

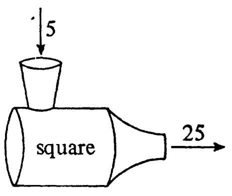

FIGURE 1.

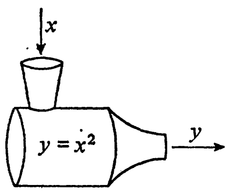

FIGURE 2.