THE PRIME NUMBER THEOREM FROM log n!

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During the nineteenth century attempts were made to prove the prime number theorem from the formula [1, pp. 87–95]

$$(1)\qquad \log n! = \sum_{p\leq n}\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots\right)\log p.$$

While remarkable good results were obtained by Chebyshev and Sylvester the prime number theorem was not obtained. That it can be obtained will be shown below but only by using the fact that $\zeta(1+iu)\neq 0$ and with the aid of Wiener's general Tauberian theorem. Thus the use of log n! seems to be no simpler than using Lambert series [2, Theorem 15] to prove the prime number theorem.

From (1) follows

$$n\log n - n + O(\log n) = \int_1^n \left[\frac{n}{y}\right]d\psi(y),$$

where

$$\psi(x) = \sum_{p^m\leq x}\log p = \sum_{p\leq x}\left[\frac{\log x}{\log p}\right]\log p.$$

Hence

$$x\log x - x + O(\log x) = \int_1^x\left[\frac{x}{y}\right]d\psi(y).$$

If

$$r(x) = [x] - x + \frac{1}{2},$$

$$(2)\qquad \begin{aligned} x\log x - x + O(\log x) &= \int_1^x r\left(\frac{x}{y}\right)d\psi(y) + x\int_1^x\frac{d\psi(y)}{y} - \frac{1}{2}\psi(x)\\ &= \int_1^x r\left(\frac{x}{y}\right)d\psi(y) + x\int_1^x\frac{\psi(y)}{y^2}\,dy + \frac{1}{2}\psi(x).\end{aligned}$$

Let

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$$g(x) = x\int_1^x \frac{\psi(y)}{y^2}\,dy + \frac{1}{2}\psi(x). \tag{3}$$

Then by (2)

$$g(x) = x\log x - x - \int_1^x r\left(\frac{x}{y}\right)d\psi(y) + O(\log x). \tag{4}$$

Let

$$H(x) = \int_1^x \frac{\psi(y)}{y^2}\,dy. \tag{5}$$

Then by (3)

$$\frac{d}{dx}\left(\frac{1}{2}x^2H(x)\right) = g(x),$$

$$\frac{1}{2}x^2H(x) = \int_1^x g(y)dy.$$

Using this with (5) in (3)

$$\psi(x) = 2g(x) - 4\int_1^x \frac{g(y)dy}{x}.$$

Using (4) this becomes

$$\psi(x) = 2x\log x - 2x - 2\int_1^x r\left(\frac{x}{y}\right)d\psi(y) + O(\log x)$$

$$-\frac{4}{x}\left\{\frac{1}{2}x^2\log x - \frac{3}{4}x^2 - \int_1^x dy\int_1^y r\left(\frac{y}{t}\right)d\psi(t)\right\}$$

or

$$\psi(x) = x - 2\int_1^x r\left(\frac{x}{y}\right)d\psi(y) + \frac{4}{x}\int_1^x d\psi(t)\int_t^x r\left(\frac{y}{t}\right)dy + O(\log x). \tag{6}$$

But

$$\int_1^x d\psi(t)\int_t^x r\left(\frac{y}{t}\right)dy = \int_1^x d\psi(y)\,y\int_1^{x/y} r(\xi)d\xi.$$

Hence (6) becomes

(7) $$\int_1^x \left\{1 + 2r\left(\frac{x}{y}\right) - 4\frac{y}{x}\int_1^{x/y} r(\xi)d\xi\right\} d\psi(y) = x + O(\log x).$$

Let

(8) $$R(x) = 1 + 2r(x) - 4\int_1^x r(\xi)d\xi/x.$$

Then since r is periodic and has average zero, $R(x)$ is bounded. It is easy to show $R(x) \geqq 0$. From (7) and (8)

$$\int_1^x R\left(\frac{x}{y}\right) d\psi(y) = x + O(\log x).$$

From this follows

$$\int_1^t \frac{dx}{x}\int_1^x R\left(\frac{x}{y}\right) d\psi(y) = t + O(\log^2 t)$$

or

$$\int_1^t d\psi(y)\int_y^t R\left(\frac{x}{y}\right)\frac{dx}{x} = t + O(\log^2 t).$$

Hence

$$\int_1^t d\psi(y)\int_1^{t/y} \frac{R(\xi)}{\xi} d\xi = t + O(\log^2 t).$$

Replacing t by x and integrating by parts

(9) $$\int_1^x \frac{\psi(y)}{y} R\left(\frac{x}{y}\right) dy = x + O(\log^2 x).$$

That $m(y) = \psi(y)/y$ is bounded follows in an elementary way from Chebyshev's inequality for $\theta(x)/x$ where

$$\theta(x) = \sum_{p \leq x} \log p.$$

If $x = e^s$ and $y = e^t$ then (9) becomes

(10) $$\int_0^s R(e^{s-t})e^{-(s-t)}m(e^t)dt = 1 + O(s^2e^{-s}).$$

If $K(s) = 0$ for $s < 0$ and

(11) $$K(s) = R(e^s)e^{-s}, \qquad s > 0$$

then since R is bounded,

$$\int_{-\infty}^{\infty} | K(s) | ds < \infty$$

and by (10)

$$\int_0^s K(s-t)m(e^t)dt = 1 + O(s^2 e^{-s}).$$

If

$$\int_0^{\infty} K(s)e^{-ius}ds = k(u) \neq 0$$

and if $k(0)=1$ it follows from Wiener's general Tauberian theorem [**2**, Theorem 4] that for any $K_1(s) \in L(-\infty, \infty)$

$$\lim_{s\to\infty} \int_0^{\infty} K_1(s-t)m(e^t)dt = \int_0^{\infty} K_1(s)ds.$$

In particular let

$$\begin{aligned} K_1(s) &= e^{-s}, \qquad s > 0, \\ &= 0, \qquad s < 0. \end{aligned}$$

Then

$$\lim_{s\to\infty} \int_0^s e^{-(s-t)}m(e^t)dt = 1,$$

or

$$\lim_{x\to\infty} \frac{1}{x}\int_1^x \frac{\psi(y)}{y}\, dy = 1.$$

It is an immediate consequence of this that, for any $\lambda > 0$,

$$\lim_{x\to\infty} \frac{1}{\lambda x}\int_x^{x(1+\lambda)} \frac{\psi(y)}{y}\, dy = 1,$$

and this leads easily to $\lim_{x\to\infty} \psi(x)/x = 1$. Thus to complete the proof of the prime number theorem it remains to show that $k(u) \neq 0$, $k(0)=1$.

In terms of R using (11)

$$k(u) = \int_1^{\infty} R(x)x^{-iu-2}dx.$$

Hence by (8)

$$k(u) = \int_1^\infty x^{-iu-2}dx + 2\int_1^\infty r(x)x^{-iu-2}dx$$
$$- 4\int_1^\infty x^{-iu-3}dx\int_1^x r(t)dt$$
$$= \frac{1}{1+iu} + 2\int_1^\infty r(x)x^{-iu-2}dx\left(1 - \frac{2}{2+iu}\right)$$
$$= \frac{1}{1+iu} + \frac{2iu}{2+iu}\int_1^\infty r(x)x^{-iu-2}dx.$$

Setting $u=0$, it follows that $k(0)=1$. Recalling the periodicity of $r(x)$

$$\int_1^\infty r(x)x^{-iu-2}dx = \sum_{n=1}^\infty \int_0^1 (\tfrac{1}{2} - x)(n+x)^{-iu-2}dx$$
$$= \lim_{\lambda\to 0} \sum_{n=1}^\infty \int_0^1 (\tfrac{1}{2} - x)(n+x)^{-iu-2-\lambda}dx.$$

Clearly

$$\int_0^1 (\tfrac{1}{2} - x)(n+x)^{-iu-2-\lambda}dx$$
$$= \frac{1}{2}\,\frac{1}{iu+1+\lambda}\left(\frac{1}{(n+1)^{1+\lambda+iu}} + \frac{1}{n^{1+\lambda+iu}}\right)$$
$$+ \frac{1}{(iu+1+\lambda)(iu+\lambda)}\left(\frac{1}{(n+1)^{\lambda+iu}} - \frac{1}{n^{\lambda+iu}}\right).$$

Hence

$$\sum_1^\infty \int_0^1 (\tfrac{1}{2} - x)(n+x)^{-iu-2-\lambda}dx = \frac{1}{iu+1+\lambda}\left\{\zeta(1+\lambda+iu) - \tfrac{1}{2}\right\}$$
$$- \frac{1}{(iu+1+\lambda)(iu+\lambda)}.$$

Thus

$$k(u) = \frac{1}{1+iu} + \frac{2iu}{(2+iu)(iu+1)}\left\{\zeta(1+iu) - \tfrac{1}{2}\right\} - \frac{2}{(2+iu)(iu+1)}$$
$$= \frac{2iu}{(2+iu)(1+iu)}\zeta(1+iu).$$

Hence $k(u) \neq 0$.

ADDENDUM (March 8, 1963). It was pointed out to me by H. R. Pitt that a different proof of the prime number theorem based on log $n!$ and using Wiener's theorem was given by A. E. Ingham, *Some Tauberian theorems connected with the prime number theorem*, J. London Math. Soc. 22 (1945), 161–180.

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FLAG-TRANSITIVE PLANES OF EVEN ORDER[1]

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1. Introduction. In a projective plane a configuration consisting of a line and a point incident with that line is called a *flag*. A collineation group which is transitive on the flags of a projective plane is called *flag-transitive* (or "acutely transitive"). Such a group is also called *sharply flag-transitive* (or "flag-regular" or "acutely regular") if in addition, the only collineation leaving a flag fixed is the identity. A projective plane is called *flag-transitive* or *sharply flag-transitive*, respectively, if it admits a group of collineations which is flag-transitive or sharply flag-transitive, respectively. D. G. Higman and J. E. McLaughlin proved the following theorem [6, Proposition 10, p. 391].

THEOREM. *Given a finite projective plane of odd order n with a flag-transitive group G where n is not a square or else $n=m^2$ and $m\equiv -1$* (mod 4). *Then the plane is Desarguesian and the group G is doubly transitive.*

In this paper, this theorem is extended to finite projective planes of many orders n. It is shown (Theorem 1) that for any integer n such that either $n+1$ or n^2+n+1 is a prime, a flag-transitive plane of order n must be Desarguesian or sharply flag-transitive. However, as

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