$$D_t T_2(e^{t/2}) = \sum_{m=0}^{\infty} me^{mt}/(m!)^2 = K_0 y'_{0,2}(t) + K_1 y'_{1,2}(t).$$

Taking $t=0$, we obtain $y_{0,2}(0)=1$ and $y_{1,2}(0)=0$, $y'_{0,2}(0)=0$, and $y'_{1,2}(0)=1$, so that

$$K_0 = T_2(1) = 2.279\ 585\ 518 \cdots, K_1 = \sum_{m=1}^{\infty} m/(m!)^2 = 1.590\ 638 \cdots.$$

For other values of n the calculations are similar.

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MATHEMATICAL NOTES

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Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE TWIN PRIME CONSTANT

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Let $T(x)$ be the number of prime twins with first member not exceeding x. Brun [1] showed that $T(x)=O(x/\log^2 x)$, and Hardy and Littlewood [2] proved that if $T(x)\sim C(x/\log^2 x)$ for some constant C (which is a plausible conjecture with excellent empirical support from the tables of prime numbers) then

$$C = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) = 1.32032 \cdots. \tag{1}$$

In 1954, E. M. Wright [3], in his *Postscript on prime pairs*, attributed a non-rigorous probabilistic derivation of (1) to Lord Cherwell, citing a 1946 paper [4]. This reference was repeated recently in this MONTHLY [5] by Pólya. Actually, the first probabilistic derivation of (1) can be found explicitly as early as Selmer's paper [6] which appeared in 1942, and perhaps implicitly in earlier writings of Brun (cf. [7]).

The probabilistic derivation of (1) is actually simpler than the presentations of it in [3], [5], and [6]. It may be done in this manner:

By the prime number theorem, Pr(x is prime)$\sim 1/\log x$. However, the probability that $x+2$ is prime, *given* that x is prime, departs from $1/\log(x+2)$ as follows: Since x is prime, x is odd with probability 1, and $x+2$ is odd, doubling the chance of $x+2$ being prime. For any odd prime p, $x \neq p$ is *not* divisible by p with probability 1, which *decreases* the probability that $x+2$ is relatively prime to p from $(p-1)/p$ to $(p-2)/(p-1)$. Thus the probability that $x+2$ is prime departs from $1/\log(x+2)$ by the factor

$$2\prod_{p>2}\left(\frac{p-2}{p-1}\right)\Big/\left(\frac{p-1}{p}\right) = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

Thus

$$T(x)/x \sim \frac{1}{\log^2 x}\cdot 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

The question of how many primes to include in the product need never arise (as it does in the references cited), because the infinite product is clearly convergent. Thus the product can be extended over all odd primes, rather than starting with $2<p<x^{\mu}$ and letting $x\to\infty$, where x^{μ} can only be justified on seemingly mystical grounds [5] at best. (In fact, different authors use different values of μ.)

In fairness to the references cited, the question of how many primes to include in the product *is* significant in estimating $\pi(x)$, the number of primes up to x, but as shown in this note (which follows the treatment in [8]), the question can be sidestepped entirely in the probabilistic estimation of $T(x)$.

It is also interesting to observe that

$$C = 2\sum_{\text{odd } n=1}^{\infty}\frac{\mu(n)}{\phi^2(n)}, \tag{2}$$

where μ and ϕ are the Möbius and Euler functions, respectively. This formula arises in the "singular series" approach of Hardy and Littlewood. Actually, (1) may be regarded as an Euler product expansion for (2).

Also,

$$C = \frac{1}{4}\sum_{\text{odd } n=1}^{\infty}\frac{\mu(n)2^{\nu(n)}\log^2 n}{n}, \tag{3}$$

where $\nu(n)$ is the number of distinct prime factors of n. The expression (3) for C arises in the approach to twin primes followed in [8].

Another interesting constant connected with the twin primes is "Brun's sum,"

$$(\tfrac{1}{3}+\tfrac{1}{5})+(\tfrac{1}{5}+\tfrac{1}{7})+(\tfrac{1}{11}+\tfrac{1}{13})+(\tfrac{1}{17}+\tfrac{1}{19})+\cdots,$$

the sum of the reciprocals of the twin primes, which Brun showed is either convergent or finite. Selmer [6] evaluated this sum to three decimal places, omitting the pair $(1/3+1/5)$, and obtaining the value $1.368 \cdots$ for the rest, guaranteed to within 0.1 per cent.

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THE QUADRATIC SUBFIELD OF THE FIELD GENERATED BY THE p-th ROOT OF UNITY

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It is well known that the field $R(\zeta_p)$ generated by the pth root of unity ζ_p (p an odd prime) contains the field generated by $\sqrt{p}$ as its only quadratic subfield if $p\equiv 1$ (4) and the field generated by $\sqrt{(-p)}$ if $p\equiv 3(4)$. An elementary proof for this discrepancy follows from the Gauss sums. Another proof can be obtained from the known expression for the discriminant of the cyclotomic field which contains p as its only prime factor. This implies that the quadratic subfield must be generated by the square root of an integer $n\equiv 1(4)$. The following alternative proof uses no algebraic number theory.

Let $p=4n+3$. Then $p-1=4n+2=2m$ where m is odd. Since $(2, m)=1$ the field $R(\zeta_p)$, which is cyclic and of degree $p-1$, is the product of a quadratic field and of a field of odd degree which is necessarily real. Since $R(\zeta_p)$ contains complex quantities the quadratic field cannot be real.

For $p=4n+1$ we can argue, *e.g.*, as follows: Since $p-1=4n$ the field $R(\zeta_p)$ contains a biquadratic cyclic field and the quadratic subfield of such a field is known to be real.*

In a vague way, a transition between the two cases is given by the case of the integers of the form $4n$. The field $R(\zeta_{4n})$, $n>1$, is no longer cyclic and contains a real and an (actually several) imaginary quadratic field. For, the field of the 8th root of unity contains $\sqrt{(-1)}$ and $\sqrt{2}$, while any $R(\zeta_{4p})$, p an odd prime, contains $\sqrt{(-1)}$ and $\sqrt{p}$ or $\sqrt{(-p)}$, hence both.

* See, *e.g.*, H. Weber, Algebra, vol. 1 (2nd ed.), 1898, p. 699.