While his developments are often prolix and involve some inaccuracies, they have placed a considerable part of the theory of substitutions into an easily accessible form and have been a source of inspiration for many of his successors. As an instance of an inaccuracy we may cite his statement (on page 443, volume 9, of the first series of his works) that a primitive group whose degree is a prime number increased by one cannot be simply transitive. It is very easily seen that the symmetric group of degree 9 can be represented as a simply transitive primitive group of degree $84=83+1$. His enumeration of the possible orders of groups of degree 6, on page 493 of the same volume, is also far from correct; but this should perhaps not surprise us in view of the large number of errors in the published enumerations of possible substitution groups.

The preceding considerations neither prove nor disprove the justice of the claim that Cauchy is the founder of group theory even if they tend to support this view. It has been our aim to exhibit a few of the elements involved in such a question, and especially to point out that many efficient workers are needed for the development of a great subject. Mathematical subjects gain in attractiveness if we can associate with them an intelligent insight into their growth and a due appreciation of the costly heritage involved in their fundamental theorems. To this end it is desirable to associate one or more founders with each of the modern subjects.

PERFECT NUMBERS.

By T. M. PUTNAM, University of California.

The theory of numbers, probably more than any other branch of mathematics, offers problems that are very easy to state and formulate completely, but extremely difficult to solve. There are many that have baffled even trained workers in this field, who have been obliged to content themselves in many cases with but partial resolutions of the questions. These very often appear as isolated, artificial problems whose solution would apparently add very little to the main body of theory. But sometimes there is an historical interest attached, which coupled with an alluring simplicity of formulation attracts investigators toward it. There is always the possibility, too, that the pursuit of solutions of even these elusive problems may lead to the discovery of mathematical relations, or processes that are new and of much more general application than to the immediate problem to be solved.

Some such justification may be necessary for research concerning the existence or relations of perfect numbers. Indeed, Fermat was led by this problem to some of his most important theorems. It is moreover a problem of much historic interest.

A perfect number is defined to be one that is equal to the sum of all its divisors exclusive of itself. If one denotes such a number by m, and the sum of all its factors including itself by $\sigma(m)$, then the defining relation is

$$\sigma(m)=2m.$$

If $m=a^\alpha b^\beta \ldots l^\gamma$ where $a, b, \ldots, l$ are distinct primes, then

$$\sigma(m)=(1+a+a^2+\ldots+a^\alpha)(1+b+b^2+\ldots+b^\beta)\ldots(1+l+l^2+\ldots+l^\gamma).$$

Hence the defining condition becomes

$$2\,a^\alpha b^\beta \ldots l^\gamma = (1+a+\ldots+a^\alpha) \ldots (1+l+\ldots+l^\gamma).$$

Euclid showed that all numbers of the form $2^{n-1}(2^n-1)$, where 2^n-1 is a prime, are perfect numbers; but it was Euler who first showed that all even perfect numbers are necessarily of this form. These theorems are easily verified by the above defining condition.

The smallest perfect numbers given by Euclid's formula are 6 and 8, but it has so far produced only nine such numbers, owing to the difficulty of determining whether or not 2^n-1 is a prime, when n is large. The known cases are for the values $n=2$, 3, 5, 7, 13, 17, 19, 31, and 61. It is not known whether or not, this formula contains an infinite number of perfect numbers.

The theory of *even* perfect numbers is therefore fairly complete. But no *odd* perfect number has been found, nor has their existence either been proved or disproved. It is possible, however, to set up certain restrictive theorems concerning them, assuming that they do exist, which so hedge them about that the chances of any surviving are at least extremely small.

Some of the more important of these theorems will be stated below, and one new one (v. 7) will be given with proof and some applications.

1. If an odd perfect number exists it has the form $p^k A^2$, where p and k are both of the form $4n+1$, and p is a prime not dividing A. (Lucas, *Theorie des Nombres*, p. 425).

2. From (1) follows that there are no perfect numbers of the form $4h+3$.

3. Bourlet gives the following (*Nouv. Ann.*, 1896, p. 297): If the divisors of a perfect number be denoted by d_i then $\Sigma\frac{1}{d_i}=2$.

4. It follows from (3) that no divisor of a perfect number can be perfect and there must be more divisors than the least prime.

5. There are no perfect numbers of the forms $p^k a^2 b^2 \ldots l^2$ and $p^k a^4 b^4 \ldots l^4$, where $p, a, b, \ldots, l$ are primes.

6. $\frac{m}{\sigma(m)} > (1-\frac{1}{a})(1-\frac{1}{b})\dots(1-\frac{1}{l}) < 2$ (Bourlet, l. c.) From this follows that $\phi(m) < \frac{1}{2}m$, *i. e.*, less than half the numbers smaller than m are prime to it. This theorem and the next one derived from it are useful in ruling out certain types of odd numbers from consideration.

7. If we denote by r the number of distinct primes in a number m, and suppose that all these primes are greater than $\frac{r}{\log 2}+1$, the number m cannot be perfect.

To prove this take the relation given in (6):

$$\frac{m}{\sigma(m)} > \frac{1}{1+\frac{1}{a-1}} \cdot \frac{1}{1+\frac{1}{b-1}} \cdots\cdots \frac{1}{1+\frac{1}{l-1}}.$$

But by hypothesis each prime is greater than

$$\frac{r}{\log 2}+1, \text{ hence } \frac{1}{1+\frac{1}{a-1}} > \frac{1}{1+\frac{\log 2}{r}}, \text{ etc.}$$

Therefore, $\frac{m}{\sigma(m)} > \left(\frac{1}{1+\frac{\log 2}{r}}\right)^r > \left(\frac{1}{1+\frac{\log 2}{n}}\right)^n \quad (n>r).$

Hence, $\frac{m}{\sigma(m)} > \lim_{n=\infty} \left(\frac{1}{1+\frac{\log 2}{n}}\right)^n$, that is, $\frac{m}{\sigma(m)} > \frac{1}{e^{\log 2}}$, or $2m > \sigma(m)$.

Hence m cannot be perfect.

Since $\frac{1}{\log 2} = 1.45+$ one can state the theorem for working purposes in the form: A number with r distinct prime factors all of which are greater than $\frac{3}{2}r+1$ cannot be perfect.

APPLICATIONS OF (7).

a) There are no odd perfect numbers of the form a^α. Here $j=1$, $\frac{3}{2}j+1=\frac{5}{2}$, and since 3 is the smallest possible prime, the proof follows at once.

b) There are no odd perfect numbers with only two distinct prime factors. Here $\frac{3}{2}j+1=4$. The only possible type is then $3^\alpha . p^\beta$. But

$\frac{m}{\sigma(m)} > (1-\frac{1}{3})(1-\frac{1}{p})$, or $\frac{m}{\sigma(m)} > \frac{2}{3}(1-\frac{1}{p})$. But $\frac{m}{\sigma(m)} = \frac{1}{2}$, if m is perfect. Hence,

$$\tfrac{1}{2} > \tfrac{2}{3}(1-\frac{1}{p}), \text{ or } p<4.$$

Hence there are no possible solutions.

c) If an odd perfect number could exist with three distinct primes, then one of these would have to be 3 or 5; for, $\frac{3}{2}j+1=5\frac{1}{2}$. But if we suppose that the form is $m=3^{\alpha} \,.\, p^{\beta} \,.\, q^{\alpha}$, then since

$$(1-\tfrac{1}{3})(1-\frac{1}{p})(1-\frac{1}{q}) < \tfrac{1}{2}, \quad (1-\frac{1}{p})(1-\frac{1}{q}) < \tfrac{3}{4}.$$

If $p=7$ and $q=11$, then $(1-\frac{1}{p})(1-\frac{1}{q}) > \frac{3}{4}$, and larger values of p and q would also be impossible in this inequality. It follows therefore that one of them must be equal to 5. The form is then $m=3^{\alpha} \,.\, 5^{\beta} \,.\, p^{\gamma} \ldots$ The same form results if we suppose 5 present instead of 3, initially.

Since $1-\frac{1}{p} < \frac{15}{16}$ it follows that $p<16$.

The possible cases are then for $p=7$, **11**, **13**. Sylvester has shown that none of these combinations of three primes can lead to perfect numbers (*Comptes Rendus*, 1888). Hence no odd perfect number exists with less than four primes. Bourlet, investigating possible perfect numbers with four primes shows that none can exist less than 2,197,845.

This last theorem shows that if the prime factors of a number are all sufficiently large there is no possibility of it being perfect, no matter what the exponents may be. On the other hand, the third theorem expresses a relation involving all the divisors which implicitly involves the exponents of the prime factors of the number in such a way that they cannot exceed certain upper limits.

PROBLEM. Show that if an odd perfect number exists, say $p^k A^2$, then A has at least one prime factor smaller than p.

NOTE.—It may be of interest to some of our readers to know that Professor Benjamin Peirce, in 1832, *Mathematical Diary*, page 267, showed that there is no perfect number of the form a^m, $a^m b^n$, $a^m b^n c^p$, a, b, c being prime numbers greater than unity. ED. F.