MAT 211 Introduction to Linear Algebra
Spring 2011
Final exam

Problem 1. Use Gauss–Jordan elimination to find all solutions of the following system:

$$\left|\begin{array}{ccccccccc} x_1 & + & & & 2x_3 & + & 4x_4 & = & -8 \\ & & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - & x_2 & + & 3x_3 & + & 4x_4 & = & -12 \end{array}\right|$$

Solution. We write the system in matrix form and use Gauss–Jordan elimination:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right] \begin{array}{c} \\ \\ -3\cdot \mathrm{I} \\ \\ \end{array} \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right] \begin{array}{c} \\ \\ -4\cdot \mathrm{II} \\ +1\cdot \mathrm{II} \end{array} \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right] \begin{array}{l} \\ \\ \text{move to IV} \\ \text{move to III} \end{array} \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \\ \div 3 \\ \\ \end{array} \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} -4\cdot \mathrm{III} \\ +1\cdot \mathrm{III} \\ \\ \\ \end{array} \to \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We see that x_1, x_2 and x_4 are leading variables, while x_3 is a free variable. We set $x_3 = s$, and then $x_1 = -2x_3$, $x_2 = 3x_3 + 4$, and $x_4 = -2$, so therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s \\ 3s+4 \\ s \\ -2 \end{bmatrix} = s \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix}$$

Problem 2. Find the matrices of the following linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$:

1. Counterclockwise rotation through an angle of $2\pi/3$.
2. Reflection about the line $x = y$.
3. Orthogonal projection onto the line $x + 2y = 0$.

Which of these transformations are invertible?

Solution. For the first problem, we use the rotation formula:

$$\begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}.$$

For the second problem, it is convenient to introduce the basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$, where $\vec{v}_1$ is a vector on the line, and $\vec{v}_2$ is a vector orthogonal to the line:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then $T(\vec{v}_1) = \vec{v}_1$, $T(\vec{v}_2) = -\vec{v}_2$, so the matrix B of the transformation T with respect to the basis $\mathcal{B}$ is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix of T with respect to the standard basis is

$$A = SBS^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For the third problem, the solution is similar. We introduce the basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$, where $\vec{v}_1$ is a vector on the line, and $\vec{v}_2$ is a vector orthogonal to the line:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then $T(\vec{v}_1) = \vec{v}_1$, $T(\vec{v}_2) = \vec{0}$, so the matrix B of the transformation T with respect to the basis $\mathcal{B}$ is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix of T with respect to the standard basis is

$$A = SBS^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}.$$

Problem 3. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

Solution. To find the inverse, we adjoin a 3×3 identity matrix and find the reduced row echelon form of the resulting 3×6 matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ -1 \cdot \mathrm{I} \\ -1 \cdot \mathrm{I} \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} -1 \cdot \mathrm{II} \\ \\ -2 \cdot \mathrm{II} \end{array} \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \begin{array}{c} +1 \cdot \mathrm{III} \\ -2 \cdot \mathrm{III} \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

The matrix in the right hand side is the inverse:

$$\begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}^{-1}.$$

Problem 4. Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

with respect to the basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution. Let S be the change of basis matrix:

$$S = [\vec{v}_1\, \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then the matrix B of the transformation T with respect to the basis $\mathcal{B}$ is

$$B = S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix}.$$

Problem 5. Consider the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^3$.

1. Find a basis of the subspace of $\mathbb{R}^3$ consisting of all vectors perpendicular to $\vec{v}$.

2. Find an orthonormal basis of this subspace.

Solution. A vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is orthogonal to $\vec{v}$ if and only if their dot product is zero:

$$x_1 + x_2 + x_3 = 0.$$

In this equation, x_1 is a leading variable, while x_2 and x_3 are free. Setting $x_2 = s$ and $x_3 = t$, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

so the vectors

$$\vec{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

form a basis for the subspace.

To find an orthonormal basis for this subspace, we perform the Gram–Schmidt process on $\vec{w}_1$ and $\vec{w}_2$:

$$\vec{u}_1 = \frac{\vec{w}_1}{||\vec{w}_1} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \vec{w}_2^{\perp} = \vec{w}_2 - (\vec{w}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \frac{\vec{w}_2^{\perp}}{||\vec{w}_2^{\perp}||} = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

Problem 6. Find the determinant of the following matrix:

$$\begin{bmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Solution. The third element in the third row is the only non-zero element in its column, so any pattern giving a non-zero value will contain it:

$$\begin{bmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & \textcircled{5} & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Any pattern containing this 5 either contains the fourth 1 in the fourth column, or is zero. Similarly, any pattern containing these two has to contain the 1 in the lower right corner, or be zero:

$$\begin{bmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & \textcircled{5} & 6 & 7 \\ 2 & 1 & 0 & \textcircled{1} & 2 \\ 2 & 1 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

Therefore, there are two non-zero patterns, one of them having disorder 0, and the other having disorder 1:

$$\begin{bmatrix} \textcircled{5} & 4 & 0 & 0 & 0 \\ 6 & \textcircled{7} & 0 & 0 & 0 \\ 3 & 4 & \textcircled{5} & 6 & 7 \\ 2 & 1 & 0 & \textcircled{1} & 2 \\ 2 & 1 & 0 & 0 & \textcircled{1} \end{bmatrix} \to 5{\cdot}7{\cdot}5{\cdot}1{\cdot}1 = 175, \begin{bmatrix} 5 & \textcircled{4} & 0 & 0 & 0 \\ \textcircled{6} & 7 & 0 & 0 & 0 \\ 3 & 4 & \textcircled{5} & 6 & 7 \\ 2 & 1 & 0 & \textcircled{1} & 2 \\ 2 & 1 & 0 & 0 & \textcircled{1} \end{bmatrix} \to (-1)4{\cdot}6{\cdot}5{\cdot}1{\cdot}1 = -120,$$

so the determinant is $170 - 125 = 55$.

Problem 7. Let A be the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

1. Write down the characteristic equation of A.
2. Find the eigenvalues of A and their algebraic multiplicities.
3. For each eigenvalue, find a basis for the corresponding eigenspace. Find the geometric multiplicities of the eigenvalues of A.
4. Is the matrix A diagonalizable? If it is, find an invertible matrix S such that the matrix
$$D = S^{-1}AS$$
is a diagonal matrix, and find D.

Solution. The characteristic equation is

$$\det(A - \lambda \cdot I_3) = \begin{vmatrix} 4-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)(1-\lambda) - (-2)(1-\lambda)(1) =$$

$$= (1-\lambda)(\lambda^2 - 5\lambda + 6) = (1-\lambda)(2-\lambda)(3-\lambda).$$

The eigenvalues are 1, 2, and 3, all with multiplicity one.

For $\lambda = 1$, the eigenspace is

$$E_1 = \ker(A - I_3) = \ker \begin{bmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We set up the system

$$\begin{bmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving it, we get that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

so the eigenspace E_1 is spanned by

$$E_1 = \text{Span} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \text{Span} \vec{v}_1.$$

Similarly, for $\lambda = 2$ the eigenspace is

$$E_2 = \ker(A - 2I_3) = \ker \begin{bmatrix} 2 & 0 & -2 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \text{Span} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \text{Span} \vec{v}_2,$$

and for $\lambda = 3$ we get

$$E_3 = \ker(A - 3I_3) = \ker \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} = \text{Span} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \text{Span} \vec{v}_3.$$

The three eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ correspond to different eigenvalues and are therefore linearly independent, so they form an eigenbasis for A. Therefore, the matrix A is diagonalizable—if we write it in terms of its eigenbasis, we will get a diagonal matrix. Let S be the change of base matrix to this eigenbasis:

$$S = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

then the matrix

$$D = S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is diagonal, and the diagonal entries correspond to the eigenvalues of A.

Problem 8. Let P_2 denote the space of polynomials of degree less than or equal to 2. Let T from P_2 to P_2 be defined as

$$T(f) = 2f' + f''.$$

1. Show that T is a linear transformation.
2. Let $\mathcal{B} = (1, t, t^2)$ be the standard basis of P_2. Find the matrix B of T with respect to the basis $\mathcal{B}$.
3. Find a basis for the kernel of T and a basis for the image of T.
4. Write down the characteristic equation for the matrix B that you found. Find the eigenvalues of B.
5. For each eigenvalue, find a basis for the corresponding eigenspace.
6. Is the matrix B diagonalizable?

Solution.

To check that T is linear, we need to show that it preserves sums:

$$T(f_1 + f_2 = 2(f_1 + f_2)' + (f_1 + f_2)'' = 2f_1' + f_1'' + 2f_2' + f_2'' = T(f_1) + T(f_2)$$

and scalar products:

$$T(kf) = 2(kf)' + (kf)'' = 2kf' + kf'' = k(2f' + f'') = kT(f).$$

To find the matrix B, we find the $\mathcal{B}$-coordinate vectors of the images of the basis elements under T:

$$[T(1)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(t)]_{\mathcal{B}} = [2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad [T(t^2)]_{\mathcal{B}} = [4t+2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix},$$

and then put these vectors together:

$$B = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

The first column vector is zero, and the second and third are linearly independent. Therefore, the kernel of B is spanned by the vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

this corresponds to the fact that the transformation T sends all constant polynomials to zero. The image is spanned by the second and third column vectors:

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}.$$

The characteristic equation for B is

$$\det(B - \lambda I_3) = \begin{vmatrix} -\lambda & 2 & 2 \\ 0 & -\lambda & 4 \\ 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)^3,$$

there is only one eigenvalue $\lambda = 0$ with algebraic multiplicity 3. The eigenspace corresponding to a zero eigenvalue is the kernel of the matrix, and we've found that the kernel is spanned by one vector. Since there is only one linearly independent eigenvector, the matrix B is not diagonalizable.

The matrix B has the property that raising it to a sufficiently high power gives the zero matrix, in this case B^3 is the zero matrix. It turns out that this is equivalent to the characteristic polynomial being $(-\lambda)^n$. Matrices of this kind are called *nilpotent*, they are never diagonalizable.