MAT 211, Spring 2012
Solutions to Homework Assignment 9

Maximal grade for HW9: 100 points

Section 5.1. 16. (10 points) Consider the vectors

$$u_1 = (1/2, 1/2, 1/2, 1/2),\ u_2 = (1/2, 1/2, -1/2, -1/2),\ u_3 = (1/2, -1/2, 1/2, -1/2)$$

in $\mathbb{R}^4$. Find all vectors u_4 such that u_1, u_2, u_3, u_4 for an orthonormal basis in $\mathbb{R}^4$.

Answer: (1/2,-1/2,-1/2,1/2), (-1/2,1/2,1/2,-1/2).

Solution: Let $u_4 = (x_1, x_2, x_3, x_4)$. One can check that

$$u_1 \cdot u_1 = 1/4 + 1/4 + 1/4 = 1/4 = 1, \quad u_1 \cdot u_2 = 1/4 + 1/4 - 1/4 - 1/4 = 0,$$

and similarly

$$u_2 \cdot u_2 = u_3 \cdot u_3 = 1, u_1 \cdot u_3 = u_2 \cdot u_3 = 0.$$

Let us impose the condition that u_4 is orthogonal to u_1, u_2, u_3:

$$\begin{cases} 1/2x_1 + 1/2x_2 + 1/2x_3 + 1/2x_4 & = 0 \\ 1/2x_1 + 1/2x_2 - 1/2x_3 - 1/2x_4 & = 0 \\ 1/2x_1 - 1/2x_2 + 1/2x_3 - 1/2x_4 & = 0 \end{cases}$$

Let us write this in matrix notation:

$$\left(\begin{array}{cccc|c} 1/2 & 1/2 & 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & -1/2 & -1/2 & 0 \\ 1/2 & -1/2 & 1/2 & -1/2 & 0 \end{array}\right)$$

Multiply the whole matrix by 2:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{array}\right)$$

Subtract the first row from the second and third:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 & 0 \end{array}\right)$$

Swap second and third row and divide them by (-2):

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

Subtract the second and third row from the first:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

Therefore $u_4 = (x_4, -x_4, -x_4, x_4)$.

The remaining condition is that u_4 should have unit length. Since $u_4 \cdot u_4 = 4x_4^2 = 1$, we have $x_4 = \pm 1/2$.

17. (15 points) Find a basis for $W^\perp$, where $W = Span((1,2,3,4),(5,6,7,8))$.

Solution: The space $W^\perp$ is defined by the system of two linear equations:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 & = 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 & = 0 \end{cases}$$

In matrix notation:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \end{array}\right)$$

Subtract the first row (multiplied by 5) from the second:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -4 & -8 & -12 & 0 \end{array}\right)$$

Divide second row by (-4):

$$\begin{pmatrix} 1 & 2 & 3 & 4 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \end{pmatrix}$$

Subtract the second row, multiplied by 2, from the first:

$$\begin{pmatrix} 1 & 0 & -1 & -2 & | & 0 \\ 0 & 1 & 2 & 3 & | & 0 \end{pmatrix}$$

Therefore x_3, x_4 are free parameters, $x_1 = x_3 + 2x_4, x_2 = -2x_3 - 3x_4$. The basis in the space of solutions: $(1, -2, 1, 0), (2, -4, 0, 1)$.

25. a) (5 points) Consider a vector v and a scalar k. Show that $||k \cdot v|| = |k|||v||$.

b) (5 points) Show that if v is a nonzero vector in $\mathbb{R}^n$, then $u = v/||v||$ is a unit vector.

Solution: a)

$$||k \cdot v||^2 = (k \cdot v) \cdot (k \cdot v) = k^2(v \cdot v) = k^2||v||^2.$$

Therefore

$$||k \cdot v|| = \sqrt{k^2||v||^2} = |k|||v||.$$

b) Let us apply (a) to $k = 1/||v||$:

$$||u|| = ||k \cdot v|| = |k|||v|| = ||v||/||v|| = 1.$$

Therefore u is a unit vector.

26. (20 points) Find the orthogonal projection of $v = (49, 49, 49)$ onto the subspace of $\mathbb{R}^3$ spanned by $v_1 = (2, 3, 6)$ and $v_2 = (3, -6, 2)$.

Answer: $(19, 39, 64)$.

Solution: Remark that

$$v_1 \cdot v_1 = 2^2 + 3^2 + 6^2 = 49, \quad v_1 \cdot v_2 = 2 \cdot 3 - 3 \cdot 6 + 2 \cdot 6 = 0, \quad , v_2 \cdot v_2 = 3^2 + (-6)^2 + 2^2 = 49.$$

Therefore v_1 and v_2 are perpendicular, and both have lengths $\sqrt{49} = 7$. Therefore the vectors

$$u_1 = \frac{1}{7}v_1 = (2/7, 3/7, 6/7), \quad u_2 = \frac{1}{7}v_2 = (3/7, -6/7, 2/7)$$

form an orthonormal basis of the subspace.

We have

$$v \cdot u_1 = 14 + 21 + 42 = 77, \quad v \cdot u_2 = 21 - 42 + 14 = -7,$$

so the projection of v can be found by a formula

$$v_{||} = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 = 77u_1 - 7u_2 = (22, 33, 66) - (3, -6, 2) = (19, 39, 64).$$

28. (20 points) Find the orthogonal projection of $v = (1, 0, 0, 0)$ onto the subspace in $\mathbb{R}^4$ spanned by $v_1 = (1, 1, 1, 1), v_2 = (1, 1, -1, -1), v_3 = (1, -1, -1, 1)$.

Answer: $(3/4, 1/4, -1/4, 1/4)$.

Solution: Similarly to the previous problem one can check that v_1, v_2, v_3 are pairwise orthogonal and $||v_1|| = ||v_2|| = ||v_3|| = \sqrt{4} = 2$. Therefore one can choose an orthonormal basis in the subspace:

$$u_1 = \frac{1}{2}v_1 = (1/2, 1/2, 1/2, 1/2), u_2 = \frac{1}{2}v_2 = (1/2, 1/2, -1/2, -1/2),$$

$$u_3 = \frac{1}{2}v_3 = (1/2, -1/2, -1/2, 1/2).$$

We have:

$$v \cdot u_1 = 1/2, \quad v \cdot u_2 = 1/2, \quad v \cdot u_3 = 1/2,$$

so

$$v_{||} = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + (v \cdot u_3)u_3 =$$

$$= 1/2(u_1 + u_2 + u_3) = 1/2(3/2, 1/2, -1/2, 1/2) = (3/4, 1/4, -1/4, 1/4).$$

29. (15 points) Consider the orthonormal vectors u_1, u_2, u_3, u_4, u_5 in $\mathbb{R}^{10}$. Find the length of the vector

$$x = 7u_1 - 3u_2 + 2u_3 + u_4 - u_5.$$

Solution: Since u_i are orthonormal, we have

$$||x||^2 = 7^2 + (-3)^2 + 2^2 + 1^2 + (-1)^2 = 64, \quad ||x|| = 8.$$

33. (10 points) Among all the vectors in $\mathbb{R}^n$ whose components add up to 1, find the vector of minimal length.

Answer: $(1/n, \ldots, 1/n)$.

Solution: We have to project the origin onto the hyperplane consisting of all vectors with sum of coordinates 1. It is clear that the projection does not change if we permute the coordinates, so all its coordinates should be equal to each other. Since their sum is 1, the desired vector is $(1/n, \ldots, 1/n)$.