MAT 211, Spring 2012
Solutions to Homework Assignment 6

Maximal grade for HW6: 100 points

Section 3.1. Describe the kernel of a matrix.

2. (10 points)

$$A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}.$$

Solution: We have a system of linear equations with the augmented matrix

$$\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 6 & 9 & 0 \end{array}\right).$$

Subtract the first row, multiplied by 3, from the second row:

$$\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Divide the first row by 2:

$$\left(\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Therefore x_2 is arbitrary, $x_1 = -\frac{3}{2}x_2$.

6. (10 points)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution: We have a system of linear equations with the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right).$$

Subtract the first row from the second and the third:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore x_2 and x_3 are arbitrary, $x_1 = -x_2 - x_3$.

10. (10 points)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution: We have a system of linear equations with the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

Subtract the third row from the second and the first:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

Subtract the second row (multiplied by 2) from the first row:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

Therefore x_3 is arbitrary, $x_1 = x_3, x_2 = -2x_3, x_4 = 0$.

Describe the image of the transformation geometrically.

18. (10 points)

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}.$$

Solution: We have a system of linear equations with the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 4 & y_1 \\ 3 & 12 & y_2 \end{array}\right).$$

Subtract the first row, multiplied by 3, from the second row:

$$\left(\begin{array}{cc|c} 1 & 4 & y_1 \\ 0 & 0 & y_2 - 3y_1 \end{array}\right)$$

The system has a solution, if $y_2 - 3y_1 = 0$, so the image is a line defined by the equation $y_2 = 3y_1$.

22. (10 points)

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{pmatrix}.$$

Solution: We have a system of linear equations with the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & y_1 \\ 3 & 4 & 2 & y_2 \\ 6 & 5 & 7 & y_3 \end{array}\right).$$

Subtract the first row, multiplied by 3/2, from the second row, and subtract the first row, multiplied by 3, from the third row:

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & y_1 \\ 0 & 5/2 & -5/2 & y_2 - \frac{3}{2}y_1 \\ 0 & 2 & -2 & y_3 - 3y_1 \end{array}\right).$$

Divide the second row by 5/2, and the third by 2:

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & y_1 \\ 0 & 1 & -1 & \frac{2}{5}y_2 - \frac{3}{5}y_1 \\ 0 & 1 & -1 & \frac{1}{2}y_3 - \frac{3}{2}y_1 \end{array}\right).$$

Subtract the second row from the third:

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & y_1 \\ 0 & 1 & -1 & \frac{2}{5}y_2 - \frac{3}{5}y_1 \\ 0 & 0 & 0 & -\frac{9}{10}y_1 - \frac{2}{5}y_2 + \frac{1}{2}y_3 \end{array}\right).$$

The system has a solution, if $-\frac{9}{10}y_1 - \frac{2}{5}y_2 + \frac{1}{2}y_3 = 0$. This is a plane in $\mathbb{R}^3$.

25. (10 points) Describe the image and kernel of the rotation through an angle of $\pi/4$ on the counter-clockwise direction on the plane.

Solution: The rotation is an invertible transformation since the inverse transformation is a clockwise rotation by the same angle. Therefore $Ker(T) = 0, Im(T) = \mathbb{R}^2$.

37. (20 points) For the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

describe the images and kernels of the matrices A, A^2, A^3 geometrically.

Solution: We have

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$A(x_1, x_2, x_3) = (x_2, x_3, 0), \quad A^2(x_1, x_2, x_3) = (x_3, 0, 0), \quad A^3(x_1, x_2, x_3) = (0, 0, 0),$$

so $Ker(A)$ is a line $x_2 = x_3 = 0$, $Im(A)$ is a plane $x_3 = 0$; $Ker(A^2)$ is a plane $x_3 = 0$, $Im(A^2)$ is a line $x_1 = x_2 = 0$; $Ker(A^3) = \mathbb{R}^3, Im(A^3) = 0$.

Section 3.2. Determine if a set is a linear subspace in $\mathbb{R}^3$.
1. (10 points) $\{(x, y, z) | x + y + z = 1\}$.

Solution: It is not a linear subspace, since it does not contain a zero vector $(0, 0, 0)$.

3. (10 points) $\{(x + 2y + 3z, 4x + 5y + 6z, 7x + 8y + 9z)\}$.

Solution: It is a linear subspace, since

$$(x + 2y + 3z, 4x + 5y + 6z, 7x + 8y + 9z) = x(1, 4, 7) + y(2, 5, 8) + z(3, 6, 9),$$

so the set of all such vectors for all possible values of x, y, z can be described as a linear span of vectors $(1, 4, 7), (2, 5, 8), (3, 6, 9)$.