MAT 211, Spring 2012
Solutions to Homework Assignments 4 and 5

Maximal grade for HW4: 100 points;
Maximal grade for HW5: 100 points

Section 2.1 (HW 5) 13. (20 points) Prove the following facts:

a) The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.

Solution: Suppose that $a \neq 0$, then we can divide the first row by a :

$$\begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix}$$

Subtract from the second row the first multiplied by c:

$$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - c\frac{b}{a} \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix}$$

This matrix has rank 2, if $ad - bc \neq 0$, and rank 1 otherwise.

If $a = 0$, we can swap two rows:

$$\begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$

This matrix has rank 2 if and only if neither b nor c equals to 0. This is equivalent to the condition $ad - bc \neq 0$.

b) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Solution: Suppose that $a \neq 0$, let us find the inverse matrix using the Gauss-Jordan elimination. We start from

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right),$$

divide the first row by a:

$$\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right),$$

subtract from the second row the first multiplied by c:

$$\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - c\frac{b}{a} & -c\frac{1}{a} & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right),$$

multiply the second row by $\frac{a}{ad-bc}$:

$$\left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right),$$

subtract from the first row the second multiplied by $\frac{b}{a}$:

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{b}{a} \cdot \frac{c}{ad-bc} & -\frac{b}{a} \cdot \frac{a}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right).$$

The case $a = 0$ is analogous.

14. (20 points) a) For which values of the parameter k is the matrix

$$\begin{pmatrix} 2 & 3 \\ 5 & k \end{pmatrix}$$

invertible?

Answer: $k \neq 7.5$.

Solution: By the previous problem, this matrix is invertible if $2 \cdot k - 3 \cdot 5 \neq 0$, that is $k \neq \frac{15}{2} = 7.5$.

b) For which values of the parameter k are all the entries of

$$\begin{pmatrix} 2 & 3 \\ 5 & k \end{pmatrix}^{-1}$$

integers?

Answer: $k = 7$ **or** $k = 8$.

Solution: By the previous problem,

$$\begin{pmatrix} 2 & 3 \\ 5 & k \end{pmatrix}^{-1} = \frac{1}{2k-15} \begin{pmatrix} k & -3 \\ -5 & 2 \end{pmatrix}.$$

The entries of the inverse matrix are integers, if $D = 2k - 15$ divides $2, 3, 5$ and k. Since D should divide 2 and 3, it can be equal to 1 or to -1. If $2k - 15 = 1$, then $k = 8$, if $2k - 15 = -1$, then $k = 7$.

Section 2.2 (HW 4) Find the matrices of the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$.

19. (20 points) The orthogonal projection onto the $x - y$ plane.

Answer:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution: We have

$$T(e_1) = e_1 = (1,0,0), \quad T(e_2) = e_2 = (0,1,0), \quad T(e_3) = (0,0,0).$$

It rests to write these vectors in columns of the matrix.

20. (20 points) The reflection about the $x - z$ plane.

Answer:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution: We have

$$T(e_1) = e_1 = (1,0,0), \quad T(e_2) = -e_2 = (0,-1,0), \quad T(e_3) = e_3 = (0,0,1).$$

It rests to write these vectors in columns of the matrix.

21. (20 points) The rotation about the z-axis through an angle of $\frac{\pi}{2}$, counter-clockwise as viewed from the positive z-axis.

Answer:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution: We have

$$T(e_1) = e_2 = (0,1,0), \quad T(e_2) = -e_1 = (-1,0,0), \quad T(e_3) = e_3 = (0,0,1).$$

It rests to write these vectors in columns of the matrix.

22. (20 points) The rotation about the y-axis through an angle θ, counter-clockwise as viewed from the positive y-axis.

Answer:

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Solution: We have $T(e_2) = e_2$, and the standard rotation matrix for the $x - z$ plane. Note that the counter-clockwise (as viewed from the positive y-axis) rotation goes from the positive z-axis to the positive x-axis, so it is in fact clockwise in $x - z$ plane, and its matrix is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

It is a standard rotation matrix by angle $(-\theta)$.

23. (20 points) The reflection about the plane $y = z$.

Answer:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solution: We have

$$T(e_1) = e_1 = (1,0,0), \quad T(e_2) = e_3 = (0,0,1), \quad T(e_3) = e_2 = (0,1,0).$$

It rests to write these vectors in columns of the matrix.

Section 2.4 (HW 5) Decide whether the matrices are invertible. If they are, find the inverse.

2. (20 points)

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Answer: Not invertible

Solution: The rows are identical, so the matrix has rank 1. Therefore it is not invertible.

4. (20 points)

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

Answer: Invertible, inverse matrix:

$$\begin{pmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{pmatrix}.$$

Solution: Let us use the Gauss-Jordan elimination to find the inverse:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

Swap the first and the third rows:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \end{array}\right).$$

Subtract the first row from the second and third:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 & 1 & -1 \\ 0 & 2 & 0 & 1 & 0 & -1 \end{array}\right).$$

Divide the 3rd row by 2 and swap it with 2nd:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 3 & 1 & 0 & 1 & -1 \end{array}\right).$$

Subtract from the 3rd row the second multiplied by 3:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & \frac{1}{2} \end{array}\right).$$

Subtract from the 1st row the 3rd:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & \frac{1}{2} \end{array}\right).$$

8. (20 points)

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Answer: Invertible, inverse matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Solution: Let us use the Gauss-Jordan elimination to find the inverse:

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Swap the 1st and the 3rd rows:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right).$$

The left matrix is already in reduced row-echelon form.