MAT211 – Linear Spaces

- Definition
- Examples
- Subspaces
- Span, linear independence, basis, coordinates
- Coordinate transformation
- Dimension

A *linear (or vector) space* V is a set of elements endowed with two operations

- \+ addition: for each f and g in V, f+g is an element in V.
- . multiplication: For each f in V and each k in R, k.v is an element in V.

Moreover, these operation satisfy the following properties

- (f + g) + h = f + (g + h)
- f + g = g + f
- There exists a unique element in V, denoted by **0** and called the neutral element such that f + **0** = **0** + f = f
- For each f in V there exists a unique element in V denoted by -f such that f+(-f)=0.

- k. (f + g) = k.f + k.g
- (c + k). f = c. f + k.f
- c.(k.f) = (c.k).f
- 1.f=f

EXAMPLES of Linear Spaces:

- R^n.
- The set of all polynomials.
- The set of all polynomials of degree at most two.
- The set of all infinite sequences of real numbers.
- The set of all m x n matrices
- The space of 2 x 2 matrices $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ such that a+d=0

Questions: In each of the linear spaces

- give examples of + and .
- What is **0** ?
- If v is in the linear space, what is -v?

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- \+ addition: for each f and g in V, f+g is an element in V.
- . multiplication: For each f in V and each k in R, k.v is an element in V.

- We say that an element f of a linear space V is a *linear combination* of the elements $f_1, f_2, \ldots, f_n$ of V if there exists scalars such that
- $f = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$

- If V is the space of 2 x 2 matrices such that a+d=0, show that

$\begin{vmatrix} 2 & -2 \\ -2 & -2 \end{vmatrix}$ is a linear combination of $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ and $\begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix}$

Is the polynomial $x^2 + x + 1$ a linear combination of $x^2 + 1$, $x^2 - 1$ and 3x+3?

A subset W of a linear space V is a *subspace* if

- **0** is in W (**0** is the "zero vector" in V).
- If f and g are in W, so is f+g.
- If f is in W and k is a scalar then k.f is in W.

Let V be the space all of 2 x 2 matrices

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Show that subset W of all the matrices such that a+d=0 is a subspace of V.

EXAMPLES

- Is the set of all 2 x 2 invertible matrices a subspace of the linear space formed by all 2 x 2 matrices?
- Denote by P_4 the set of all polynomials of degree at most 4. Is P_2 a subspace of P_4?
- Is the subset of all polynomials of degree 2 a subset of P_4?

Consider the elements $f_1, f_2, \ldots, f_n$ in a linear space V

- $f_1, f_2, \ldots, f_n$ *span V* if every element in V is a linear combination of the elements $f_1, f_2, \ldots, f_n$.
- f_i is *redundant* if it a linear combination of t $f_1, f_2, \ldots, f_{i-1}$
- $f_1, f_2, \ldots, f_n$ are *linearly independent* if none of them is redundant.
- $f_1, f_2, \ldots, f_n$ form a *basis* if they are linearly independent and span V.

Example: Consider the linear space M of all matrices 2x3.

a. Find elments $f_1, f_2, \ldots, f_n$ in M that span M
b. Find a basis of M.
c. Can you find a basis of M that does not span?
d. Can you find a subset of M that spans but it is not a basis? If so, indicate the redundant vectors.

Suppose that the elements $f_1, f_2, \ldots, f_n$ are a basis of a vector space V.

- Any element f in V can be written as $c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$ for some scalars $c_1, c_2, \ldots, c_n$
- The coefficients $c_1, c_2, \ldots, c_n$ are called the *coordinates of f with respect to the basis B*= $(f_1, f_2, \ldots, f_n)$
- The vector $[c_1, c_2, \ldots, c_n]$ in R^n is called the *coordinate vector* of f and denoted by $[f]_B$.
- The transformation L: V -> R^n, defined by L(f)=[f] is a linear transformation *called the B-coordinate transformation.*

- Find two different bases of P_2.
- Find the coordinates of the polynomial (x-1)(x+1) with respect to each of these two bases.

Decribe the B-coordinate transformation $L:P_2 \to R^n$, where B is the basis $(1,x,x^2)$

If f=(x-1)(x+1) and $g=x^2 -3x+ \pi$, check that

- L(f+g)=L(f)+L(g)
- L(k.f)=k.L(f).

Theorem: If a basis of a vector space has n elements then all basis of a linear space have the n elements.

Definition: If a linear space V has a basis with n elements we say that the dimension of V is n.

EXAMPLE: Find a basis and determine the dimension

- The linear space of all 2 x 2 matrices.
- The space of all 2 x 2 matrices such that a+d=0.
- P_2 , the space of all polynomials of degree at most 2.
- The space of all polynomials.
- The space of all 2 x 2 matrices that commute with

$$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$