MAT 211

- The matrix of a linear transformation
- The B-matrix of a linear transformation
- The columns of the B-matrix of a linear transformation
- Change of basis matrix
- Change of basis in a subspace of R^n
- Change of basis for the matrix of a linear transformation

Checklist 4.1 and 4.2

- 4.1 Linear spaces.
  - Definition (when a set is a linear space)
  - Main examples
    - R^n
    - Matrices
    - Polynomials
  - Vectors $f_1, f_2, .., f_n$ in a linear space V
    - span
    - linear independent,
    - form a basis.
  - Coordinate transformation
  - Dimension of a linear space

- 4.2 Linear transformations and isomorphisms
  - Definition (when a function is a linear transformation.)
- Kernel, image, rank, nullity of a linear transformation. (what are they and how to compute them)
- Rank-nullity theorem
- Isomorphism
  - Definition
  - Properties (what is the kernel, image, dimension of domain and target space)

EXAMPLE

Consider the space U of upper triangular 2 x 2 matrices and the linear transformation T from U to U defined by T(M)=AM where A is

$$\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$$

Find a basis B of U and for each element z of U, find $[T(z)]_B$

The B-matrix of a linear transformation

- Consider a linear transformation T from V to V where V is an n-dimensional linear space. Let **B** denote a basis of V.
- The matrix A of the transformation from R^n to R^n defined by L_B o T o L_B^{-1} is called the ***B**-matrix of T.*
- If A is the **B**-matrix of T then $A.[v]_B=[T(v)]_B$.
- The columns of B are the B-coordinate vectors
  $[T(b_1)]_B, T(b_2)]_B, .., [T(b_n)]_B$

- Consider the space U of upper triangular 2 x 2 matrices and the linear transformation T from U to U defined by T(M)=AM where A is

Find a basis **B** of U and the **B**-matrix of T

$$\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix}$$

Summary of this section: A and **B** bases of linear space V, T a linear transformation from V to V

- Coordinate Transformation from V to Rn $L(f)= [f]_B$
- **B**-matrix of T is L_B o T o L_B^{-1}
- **B**=$(f_1, f_2, .., f_n)$ then **B**-matrix of T is $[[T(f_1)]_B, [T(f_2)]_B .., [T(f_n)]_B]$
- The matrix $S = S_{B\to A}$ of the linear transformation L_A o $(L_B)^{-1}$ from R^n to R^n is called the *change of basis matrix from B to A.*
- The change of basis from B= $(b_1, b_2, .., b_n)$ to A is $([b_1]_A, [b_2]_A, .., [b_n]_A)$.
- If f is in V then $[f]_A = S [f]_B$ where S is the change of basis matrix from **B** to **A**
- **A**=$(a_1, a_2, .., a_m)$ and **B**=$(b_1, b_2, .., b_m)$, S the change of basis matrix from **B** to **A**. Then $[b_1, b_2, .., b_m]= [a_1, a_2, .., a_m]S$.

The B-matrix of a linear transformation

- Consider a linear transformation T from V to V. Let B be matrix of T with respect to a basis **B**=$(b_1, b_2, .., b_n)$
- The columns of B are the B-coordinate vectors
  $[T(b_1)]_B, T(b_2)]_B, .., [T(b_n)]_B$

- Give the matrix of the linear transformation $T(f)=f''+2f'$ from P_2 to P_2 with respect to the basis $(1, t, t^2)$.
- Find basis of the kernel and the image and compute rank and nullity of T.

Change of basis matrix from B to A

- Recall: The coordinate Transformation from V to R^n is $L(f)= [f]_B$
- Consider two basis **A** and **B** of an n-dimensional vector space V.
- The matrix $S = S_{B\to A}$ of the linear transformation $L_A \circ (L_B)^{-1}$ from R^n to R^n is called the *change of basis matrix from B to A.*
- The change of basis matrix from $B= (b_1,b_2,.., b_n)$ to A is

$$([b_1]_A, [b_2]_A, .., [b_n]_A)$$

- Find the change of basis matrix S from the standard basis **B** to basis **A** of U^{2x2} where **A is**

$$\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$$

Change of basis matrix from B to A

- Recall: The coordinate Transformation from V to R^n is $L(f)= [f]_B$
- Consider two basis **A** and **B** of an n-dimensional vector space V.
- The matrix $S = S_{B\to A}$ of the linear transformation $L_A \circ (L_B)^{-1}$ from R^n to R^n is called the *change of basis matrix from B to A.*
- The change of basis matrix from $B= (b_1,b_2,.., b_n)$ to A is

$$([b_1]_A, [b_2]_A, .., [b_n]_A)$$

- $S_{B\to A}$ is an invertible matrix and $S_{B\to A} = (S_{A\to B})^{-1}$
- If f is in V then $[f]_A = S\,[f]_B$ where S is the change of basis matrix from **B** to **A**
- $\mathbf{A}=(a_1,a_2,..,a_m)$ and $\mathbf{B}=(b_1,b_2,..,b_m)$, S the change of basis matrix from **B** to **A**. Then $[b_1,b_2,..,b_m]= [a_1,a_2,..,a_m]S$.

Example: In the plane V defined by the equation 2x+y-2z=0 consider the basis

- A=((1,2,2),(2,-2,1)) and B=((1,2,2),(3,0,3))

1. Find the change of basis matrix from B to A
2. Find the change of basis matrix from A to B
3. Write an equation relating the matrices $[a_1,a_2]$ and $[b_1,b_2]$ where $A= (a_1,a_2)$ and $B= (b_1,b_2)$

Change of basis matrix from B to A

- Recall: The coordinate Transformation from V to R^n is $L(f)= [f]_B$
- Consider two basis **A** and **B** of an n-dimensional vector space V.
- The matrix $S = S_{B\to A}$ of the linear transformation $L_A \circ (L_B)^{-1}$ from R^n to R^n is called the *change of basis matrix from B to A.*
- The change of basis matrix from $B= (b_1,b_2,.., b_n)$ to A is

$$([b_1]_A, [b_2]_A, .., [b_n]_A)$$

- $S_{B\to A}$ is an invertible matrix and $S_{B\to A} = (S_{A\to B})^{-1}$
- If f is in V then $[f]_A = S\,[f]_B$ where S is the change of basis matrix from **B** to **A**
- $\mathbf{A}=(a_1,a_2,..,a_m)$ and $\mathbf{B}=(b_1,b_2,..,b_m)$, S the change of basis matrix from **B** to **A**. Then $[b_1,b_2,..,b_m]= [a_1,a_2,..,a_m]S$.

EXAMPLE: Find the matrix of a linear transformation T from P_2 to P_2 defined by $T(p(t))=p(-t)$ with respect to the basis $\mathbf{B}=(t-1, t+1, t^2)$. Determine whether T is an isomorphism and find the kernel, image, rank and nullity.

Consider the basis $\mathbf{A}=(-1, -t, t^2)$ of P_2 .

1. Find the change of basis matrix from **B** to **A**
2. Find the change of basis matrix from **A** to ***B***
3. Write an equation relating the matrices of T with respect to A and the matrix of T with respect to B