Problem 1.3.5, 4ed

- Write the system below in vector form.
  - $x + 2y = 7$
  - $3x + y = 11$
- Use your answer to represent the system and represent the solution geometrically.
- (not from the book) Write the system below in vector form.
  - $x + 2y + 4z = 7$
  - $3x + y - 2z = 11$
- Arguing geometrically, show that this system has infinitely many solutions.

Definition

A vector b in Rm is a a linear combination of the vectors $v_1, v_2, \ldots, v_n$ in Rm if there exist scalars $x_1, x_2, \ldots x_n$ such that

$b = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n$

Think of vectors as columns (in all this slides)

Problem

Determine whether the vector (1,4) is a linear combination of the vectors (5,8) and (3,5).

Think of vectors as columns (in all this slides)

MAT211

Linear Transformations

Review: Definition of Function

A function is a relation between two sets, the domain and the range such that each element of the domain is associated with at least one element of range.

1. $F(x,y) = x$
2. $G(x,y,z)=(x,x+y)$
3. $H(x,y,z)=(x^2,y)$
4. $T(x,y)=(x,y)$
5. $U(x)=\sin(x)$
6. $V(x)=2x+3$
7. $Z(x)=1/x$
8. $W(x,y)=(y+1, -10x, 2x+3y)$

Think of vectors as columns (in all this slides)

Examples

- Consider two vectors $a=(a_1,a_2)$ and $x=(x_1,x_2)$.
- The dot product a.x is a scalar.
- The assignment x->a.x (or f(x)= a.x) is a function.
- This is an example of linear function.
- What are the domain and range?

Example of function

- Consider a 2 x 2 matrix A and a vector $x=(x_1,x_2)$. (think of this vector as column)
- The product A.x is a new vector with two entries.
- The assignment x->A.x is a function.
- This is an example of linear function.

Definition

- A function f from R^m to R^n is a linear transformation if there exists an n x m matrix A such that for each x in R^m, f(x)=A.x

Which of the following functions are linear transformations?

1. F(x,y) = x
2. G(x,y,z)=(x,x+y)
3. $H(x,y,z)=(x^2,y)$
4. T(x,y)=(x,y)
5. U(x)=sin(x)
6. V(x)=2x+3
7. Z(x)=1/x
8. W(x,y)=(y+1, -10x, 2x+3y)

Find the matrices associated to each of the functions which are linear transformations.

The identity matrix and transformation

- A function f from R^n to R^n defined for each x in R^n by f(x)=x is called the identity transformation.
- Is the identity a linear transformation?
- The matrix associated with the identity transformation is the identity matrix.
- Examples
- Is the linear transformation f(x,y,z)=(x,y) the identity transformation?

Definition

- The vectors of R^n, e_1 =(1,0,0,…), e_2=(0,1,0,0,…), e_n =(0,0,…0,1) are called the standard vectors.

Example

- Find the image of the standard vectors under the linear transformation with matrix

1 2 3 1
0 4 -1 2
1 3 2 0

Theorem

- If T is a linear transformation from R^m to R^n then the matrix of T is $[T(e_1), T(e_2),\ldots,T(e_m)]$

Example

- Find the image of the standard vectors under the linear transformation with matrix

  1 2 3 1

  0 4 -1 2

  1 3 2 0

Problem

1. Consider a linear tranformation T R^2 to R^2 such that T(v)=3v and T(w)=(-1/2)w for the vectors v=(1,2) and w=(-3, 3). Sketch geometrically T(x) for a given vector x.
2. Given a vector v, and a scalar k, find the relationship between k.T(v) and T(k.v).
3. Given two vectors, v and w, find the relationship between T(v), T(w) and T(v+w).

Theorem

A function f from R^m to R^n is linear if an only if

1. For every x and y in R^m, f(x+y)=f(x)+f(y).
2. For every x in R^m and for every scalar k , f(k.x)=k.f(x).

Recall: A function f from R^m to R^n is a <u>linear transformation</u> if there exists an n x m matrix A such that for each x in R^m, f(x)=A.x

Study which of the following functions are linear transformations using the theorem below

1. F(x,y) = x
2. G(x,y,z)=(x,x+y)
3. H(x,y,z)=(x^2,y)
4. T(x,y)=(x,y)
5. U(x)=sin(x)
6. V(x)=2x+3
7. Z(x)=1/x
8. W(x,y)=(y+1, -10x, 2x+3y)

Theorem: A function f from R^m to R^n is linear if an only if

1. For every x and y in R^m, f(x+y)=f(x)+f(y).
2. For every x in R^m and for every scalar k , f(k.x)=k.f(x).

Review:

- A function g from R^n to R^m is the <u>inverse</u> of f if for each x in R^m g(f(x)) = x and for each y in R^n, f(g(y)) = y.
- Example: If f(x)=x+4 then g(y)=x-4 is the inverse of f.
- The inverse of a function f is denoted by f^{-1}.
- Note that not all functions have an inverse.
- Question: If f has an inverse, f^{-1}, does f^{-1} have an inverse? If so, what is the inverse of f^{-1}?

Find the inverse of the linear transformations below if possible.

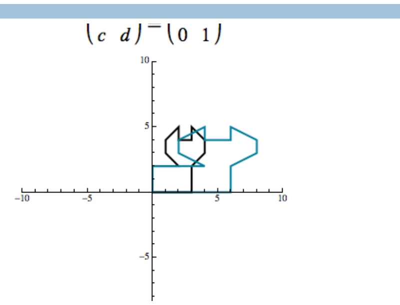
1. $F(x)=-4x$
2. $G(x,y)= (5x+3y,\ 8x+5y)$
3. $H(x,y,z)=(y,y+z,z)$

Invertible matrices

- If the linear transformation $T(x)=Ax$ is invertible, then the inverse T^{-1} is also a linear transformation. (Can you prove it?)
- Thus, there exists a matrix B such that $T^{-1}(y)=By$.
- The matrix B is the inverse of A and we write $B=A^{-1}$

Find the inverse of the matrix associated to the linear transformation G (x,y)=(5x+3y,8x+5y).

The effect of a linear transformation demo



Example

Study the effect of the linear transformations defined by the following matrices on the standard vectors. In each case, decide whether the transformation ins invertible. Find the inverse if it exists.

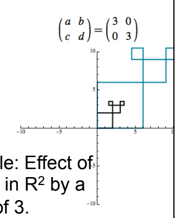
$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Definition

A linear transformation T from R^n to R^n defined by $T(x)=k.x$, is a called a <u>scaling</u>

Question: What is the effect of a scaling on a cube in R^3 ?



Example: Effect of scaling in R^2 by a factor of 3.