Review

- Subspaces of R^n
- $f:R^m \to R^n$, linear transformation, im(f) and ker(f).
- Linear combination.
- Linear independence.
- Basis and unique representation.

- Consider vectors $v_1, v_2, .., v_m$ in R^n.
- The vector v_i is <u>*redundant*</u> if v_i is a linear combination of $v_1, v_2, .., v_{i-1}$.
- The vectors $v_1, v_2, .., v_m$ are <u>*linearly independent*</u> if none of them is redundant.
- Suppose that the vectors $v_1, v_2, .., v_m$ span a subspace V. If $v_1, v_2, .., v_m$ are linearly independent we say that they <u>*form a basis*</u> of V.
- If at least one vector v is redundant then $v_1, v_2, .., v_m$ are <u>*linearly dependent*</u>.
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Theorem.

- Consider vectors $v_1, v_2, .., v_p$ and $w_1, w_2, .., w_q$ in a subspace V of R^n. If the vectors $v_1, v_2, .., v_p$ are linearly independent and the vectors $w_1, w_2, .., w_q$ span V then $q \geq p$.
- All basis of a subspace V of R^n have the same number of vectors.

Definition: The number of vectors in a basis of a subspace V of R^n is called the *dimension* of V and denoted by dim(V)

EXAMPLE

- Find a basis of the subspace V of R^3 spanned by the vectors (0,0,1), (1,1,0),(0,1,0).
- Compute the dimension of V.

Example

- Find a basis of a the line defined by the equation y=x/10.
- What is the dimension of a line in R^n?

Theorem: Consider a subspace V of R^n and $v_1, v_2, .., v_p$ vectors in V.

- If $v_1, v_2, .., v_p$ are linearly independent then $p \leq \dim(V)$
- If $v_1, v_2, .., v_p$ span V then $p \geq \dim(V)$.
- If $v_1, v_2, .., v_{\dim(V)}$ are linearly independent then $v_1, v_2, .., v_{\dim(V)}$ form a basis of V.
- If $v_1, v_2, .., v_{\dim(V)}$ span V then $v_1, v_2, .., v_{\dim(V)}$ form a basis of V.

EXAMPLE: Find a basis of the kernel and the image

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix}$$

Recall

Consider a matrix A.

A basis of im(A) can be constructed by listing the columns of A and “crossing out” the redundant vectors.

Theorem: Consider a matrix A.

- The columns of A that correspond to the columns of rref(A) containing the leading 1’s form a basis of A.
- dim(im A)=rank(A).
- If A is n x m then

dim(ker A)+ dim(im A)= m

Theorem

Consider a matrix A with columns $v_1, v_2,..v_n$.

Suppose that v_i is a redundant vector. Write $v_i = c_1 v_1 + c_2 v_2 +..+c_{i-1} v_{i-1}$ then the following vector is in the basis of ker(A).

$$\begin{bmatrix} -c_1 \\ -c_2 \\ .. \\ -c_{i-1} \\ 1 \\ 0 \\ .. \\ 0 \end{bmatrix}$$

All vectors constructed in this way form a basis of ker(A).
The non-redundant columns of A form a basis of im(A).

Find a basis of the kernel and the image

$$\begin{bmatrix} 1 & -1 & 0 & 10 & 2 & 0 \\ 1 & -1 & -1 & 9 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

Theorem

The vectors $v_1, v_2,..v_n$ of R^n form a basis of R^n if and only if the matrix with columns $v_1, v_2,..v_n$ is invertible.