MAT211 Lecture 16

Orthogonal transformations and orthogonal matrices

Definition

A linear transformation from R^n to R^n is called *orthogonal* if it preserves the length vectors. In symbols,

$||T(x)||=||x||$ for all x in R^n.

The matrix A of an orthogonal transformation is said to be an *orthogonal matrix.*

Two examples:

A rotation in R^2 and a reflexion in R^n are orthogonal transformations.

Questions:

- Are projections orthogonal transformations?
- What is the kernel of an orthogonal transformation?

Example: Determine whether the matrices are orthogonal

$$\frac{1}{\sqrt{3}}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{vmatrix} \qquad \begin{vmatrix} 0 & 1/\sqrt{2} \\ 1 & 1/\sqrt{2} \end{vmatrix}$$

Theorem

An orthogonal transformation T in R^n preserves angles; that is for each x and y in R^n the angle between x and y equals the angle between T(x) and T(y).

Question: If a transformation preserves angles, is it orthogonal?

Theorem

A linear transformation from R^n to R^n is orthgonal if and only if the vectors ($T(e_1)$, $T(e_2)$,.., $T(e_n)$) form an orthonormal basis.

A matrix A is orthogonal if an only if the columns of A form an orthonormal basis.

Example: Determine whether the matrices are orthogonal

$$\frac{1}{\sqrt{3}}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{vmatrix} \qquad \begin{vmatrix} 0 & 1/\sqrt{2} \\ 1 & 1/\sqrt{2} \end{vmatrix}$$

Theorem

The product of orthogonal matrices is orthogonal.

The inverse of an orthogonal matrix is orthogonal.

Definition

The *transpose* of an m x n matrix A, denoted A^t, is the n x m matrix which contains in the i,j entry the j,i entry of A.

An n x n matrix is *symmetric* if $A=A^t$.

An n x n matrix is *skew-symmetric* if $A=-A^t$.

Example: Find $A^{t.}$ Determine whether A is symmetric. Determine whether A is skew-symmetric

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} \qquad \begin{vmatrix} 0 & 1/\sqrt{2} \\ 1 & 1/\sqrt{2} \end{vmatrix}$$

Example

Consider W the subset of 3x3 matrices, formed by the skew-symmetric matrices. Is it a subspace of all 3 x 3 matrices? If so, what is the dimension?

(What about the same problem with symmetric instead of skew-symmetric matrices)

Theorem

- If v and w are column vectors in R^n then the dot product of v and w equals the matrix product $v.w^t$.
- An n x n matrix is orthogonal if and only if $A.A^t = I_n$.

Example: Determine whether the matrices are orthogonal

$$\begin{vmatrix} 1/\sqrt{3} & 1/\sqrt{2} & \frac{1}{2} \\ 1/\sqrt{3} & /\sqrt{2} & \frac{1}{2} \\ 1/\sqrt{3} & 0 & -1/\sqrt{2} \end{vmatrix} \qquad \begin{vmatrix} 0 & 1/\sqrt{2} \\ 1 & 1/\sqrt{2} \end{vmatrix}$$

Theorem

- If A is an n x p matrix, and B is an p x m matrix then $(A.B)^t=B^t.A^t$.
- If A is an orthogonal matrix then $A^{-1}=A^t$.
- If a is an n x n invertible matrix, then A^t is also invertible and $(A^t)^{-1}=(A^{-1})^t$.
- For any matrix A, rank(A)=rank(A^t)

Theorem

If u_1, u_2,..u_m is an orthonornal basis of a subspace V of R^n then the matrix of the projection onto V is $Q.Q^t$ where Q is the matrix with columns u_1, u_2,..u_m

Review

An n x n matrix is orthogonal if and only if $A.A^t = I_n$.

A matrix is *symmetric* if $A=A^t$.

A matrix is *skew-symmetric* if $A=-A^t$.

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Find orthogonal transformation T from R3 to such that T(⅔,⅔,⅓)=(0,0,1)

Find the matrix of the orthogonal projection of the line in R^n spanned by the vector. (1,1...1)

Given an example of a non-zero skew symmetric matrix A and compute A^2

Let A be the matrix of an orthogonal projection. Find A^2 in two ways

Geometrically

Using the formula we saw.