MAT211 Lecture 11

- •Definition Linear Spaces
- Examples
- •Subspaces
- •Span, linear independence, basis, coordinates
- •Coordinate transformation
- Dimension
- •Differential equation Solution space

Definition

- A <u>linear (or vector) space</u> V is a set of elements endowed with two operations
- + addition: for each f and g in V, f+g is an element in V.
- . multiplication: For each f in V and each k in R, k.v is an element in V.
- Moreover, these operation satisfy the following properties:

Definition (cont) for each f and g in V

(f + g) + h = f + (g + h)

There exists a unique element in V, denoted by **0** and called the neutral element such that f +**0** = **0** + f = f

For each f in V there exists a unique element in V denoted by -f such that f+(-f)=0.

Definition (cont) for each f and g in V, each c and k in R,

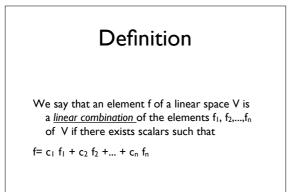
k. (f + g) = k.f + k.g (c + k). f = c. f + k.f c.(k.f) = (c.k).f l.f=f

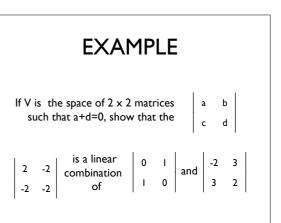
EXAMPLES of Linear Spaces:• \mathbb{R}^n .• The set of all m x n matrices• The space of 2 x 2 matrices $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ • The space of 2 number of 2 number of a linear space, what is +, ., $\mathbf{0}$, if v is in the linear space, what is -v?

MORE EXAMPLES of Linear Spaces:

- The set of all infinite sequences of real numbers. (addition and mult. are defined term by term)
- F(R,R) the set of all functions from R to R.
- The set of all polynomials of degree n at most n.
- Set of geometric vectors in plane.

Question: What is +, ., $\mathbf{0}$, if v is in the linear space, what is -v?





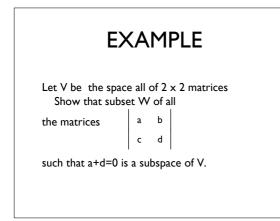
EXAMPLE

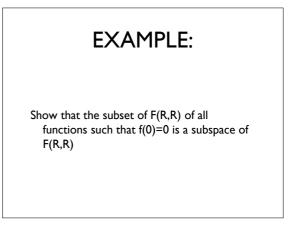
Is the polynomial $x^2 + x + 1$ a linear combination of $x^2 + 1$, x^2-1 and 3x+3?

Definition

A subset W of a linear space V is a subspace if

- W contains the neutral element of V.
- If f and g are in W, so is f+g.
- If f is in W and k is a scalar then k.f is in W.





EXAMPLE

- Is the set of all 2 x 2 invertible matrices a subspace of the linear space formed by all 2 x 2 matrices?
- Denote by P_n the set of all polynomials of degree at most n. Is P₂ a subspace of P_n?
- Is the subset of all polynomials of degree 2 a subset of P_n ?

Definition: Consider the elements f_1 , f_2 ,..., f_n in V

- f₁, f₂,...,f_n <u>span V</u> if every element in V is a linear combination of the elements f₁, f₂,...,f_n.
- f_i is <u>redundant</u> if it a linear combination of t f_1 , $f_2,...,f_{i-1}$
- f₁, f₂,...,f_n are <u>linearly independent</u> if none of them is redundant.
- f₁, f₂,...,f_n form a <u>basis</u> if they are linearly independent and span V.

Example

Consider the linear space M of all matrices 2x3.

Find a finite set that span M

Find a basis of M.

Can you find a basis that does not span?

Can you find a subset that span but it is not a basis? If so, indicate the redundant vectors.

Suppose that the elements f_1 , f_2 ,..., f_n are a basis of a vector space V.

Then any element f in V can be written as $c_1 f_1 + c_2 f_2 + ... + c_n f_n$ for some scalars $c_1, c_2, ..., c_n$

The coefficients $c_1, c_2, ..., c_n$ are called the <u>coordinates</u> of f with respect to the basis B= (f₁, f₂,...,f_n)

The vector $[c_1, c_2, ..., c_n]$ in \mathbb{R}^n is called the <u>coordinate vector</u> of f and denoted by $[f]_B$

EXAMPLE

Find a basis B of P_2 . Find the coordinates of the polynomial (x-1)(x+1) with respect to B.

Suppose that the elements f_1 , f_2 ,..., f_n are a basis of a vector space V.

The vector $[c_1, c_2, ..., c_n]$ in \mathbb{R}^n is called the <u>coordinate vector</u> of f and denoted by $[f]_B$

The transformation L: V $\rightarrow R^n$, defined by L(f)=[f] is called the B-coordinate transformation.

If f and g are in V and k is a scalar then

L(f+g)=L(f)+L(g) and L(k.f)=kL(f).

EXAMPLE

For the basis of P_2 we find in the previous example, describe the transformation

 $L:P_2 \rightarrow R^n$.

- If f=(x-1)(x+1) and $g=x^2-3x+\pi$, check that
- L(f+g)=L(f)+L(g)
- L(k.f)=k.L(f).

Theorem

- If a basis of a vector space has n elements then all basis of a linear space have the n elements.
- Definition: If a linear space V has a basis with n elements we say that the dimension of V is n.

EXAMPLE: Find a basis and determine the dimension:

- The linear space of all 2 x 2 matrices.
- The space of all 2×2 matrices such that a+d=0.
- P_2 , the space of all polynomials of degree at most 2.
- The space of all 2 x 2 matrices that commute with
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EXAMPLE: Find a basis and determine the dimension:

• The space of all polynomials.

Theorem

If a basis of a vector space has n elements then all basis of a linear space have the n elements.

Definition: If a linear space V has a basis with n elements we say that the dimension of V is n, and that V is finite dimensional.

