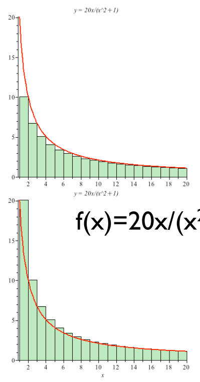
MAT 132 Series

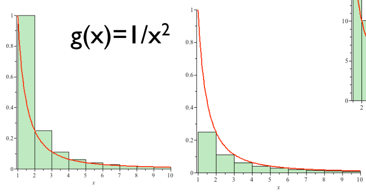
8.3 Some convergency tests and estimating sums

Consider the sequences $\{a_n\}$ and $\{b_n\}$ defined by functions $a_n = 20n/(n^2+1)$ and $b_n = 1/n^2$

Are the series associated with each of these sequences convergent or divergent?



$f(x)=20x/(x^2+1)$



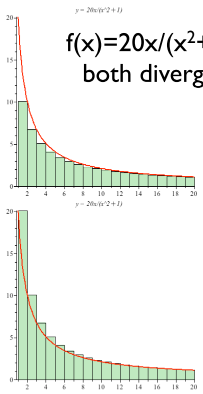
$g(x)=1/x^2$

Theorem

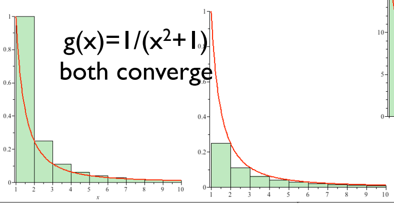
If we have a sequence defined by a function, $a_k = f(k)$ ($k = 1, 2, \ldots$) where the function f is positive, continuous, decreasing for $x \geq 1$

Then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$

both converge or both diverge



$f(x)=20x/(x^2+1)$
both diverge



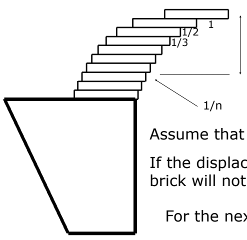
$g(x)=1/(x^2+1)$
both converge

Important Example

Use the integral test to determine wether the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

is convergent or divergent.

1 The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.



To build a cupola the builder lays bricks above each other and displaces them as far to the right as possible without making the building collapse. This is indicated in the picture on the left.

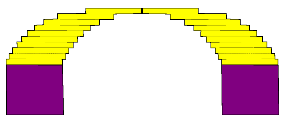
Assume that all the bricks have the length 2 units.

If the displacement of the highest brick is 1 unit, the brick will not fall.

For the next highest brick the displacement can be ½.

One can continue this inductively. The displacement of the n^{th} highest brick can be 1/n. Since the harmonic series diverges, one can build arbitrarily wide cupola in this way.

Cathedral of Florence and the Cupola of Brunelleschi



Since the harmonic series diverges, one could build arbitrarily large cupola whose stability is based on this fact. Prime example of a large cupola is the cupola of the cathedral of Florence. This cupola was designed by Brunelleschi. The cupola was completed in 1436. The shape and the width of the cupola of Brunelleschi suggest, however, that its stability is not based on the divergence of the harmonic series.

Examples to remember:

- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if p>1.
- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if p≤1.
- The geometric series $\sum_{n=1}^{\infty} a\, r^n$ is convergent if -1<r<1.
- The geometric series $\sum_{n=1}^{\infty} a\, r^n$ is convergent if -1<r<1.

Example Determine whether the series below are convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{3^n+2} \qquad \sum_{n=1}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+\pi}$$

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{3}n\pi\right)}{n^2+2}$$

The Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.
(a) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.
(b) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

Example Determine whether the series below are convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \qquad \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Definition of the remainder after k terms.

- If an infinite series is convergent and the sum is S, then the remainder after k terms, obtained by approximating the sum of the series by the k-th partial sum s_k, is denoted by R_k and is $R_k = S - s_k$.
- Example: Find R_3 for the geometric series $\sum_{n=1}^{\infty} \frac{4}{5^n}$
  - If we approximate the series by s_3, what is the error?

Example

- Approximate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the first 4 terms.
- Estimate the error of the approximation.
- How many terms are required to ensure that the sum is accurate to within 0.0001?
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3 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\, dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\, dx$$