MAT132

Directions fields and Euler's method

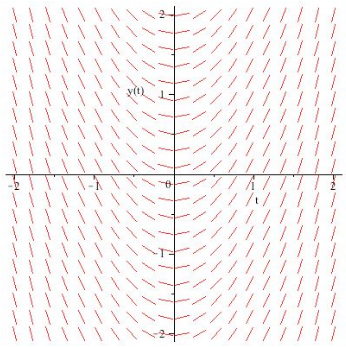
Problem

- Consider the differential equation y'=2x
- Suppose that y=y(x) is a solution.
- What can be say about the slope of the tangent to curve {(x,y(x))} for different values of x?

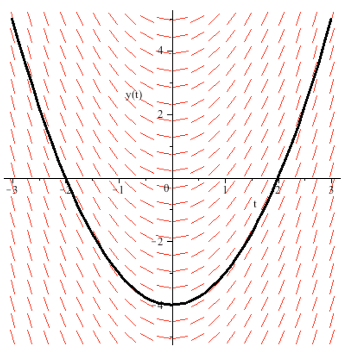
Problem

- Consider the differential equation y'=2x
- For each point (x_0 ,y_0) of the plane, draw a short piece of a line passing through (x_0 ,y_0) with slope 2 x_0

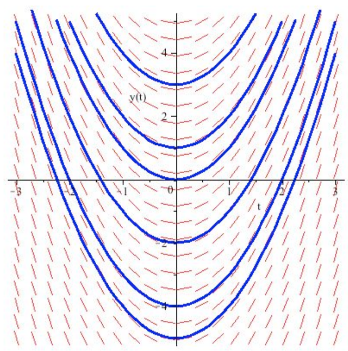
Direction field for y'=2x



Solution of y'=2x, y(0)=-4



Some solutions of y'=2x

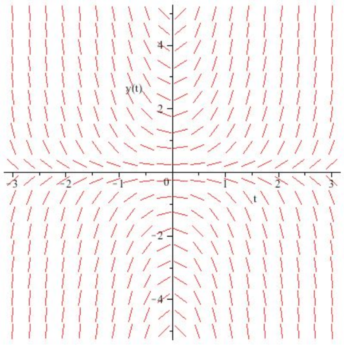


- Does this equation have equilibrium solutions?

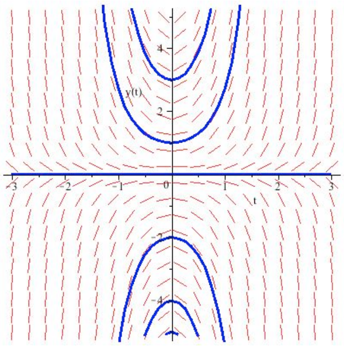
Problem

- Consider the differential equation y'=2.x.y
- For each point (x_0, y_0) of the plane, draw a short piece of a line passing through (x_0, y_0) with slope $2\, x_0\, y_0$

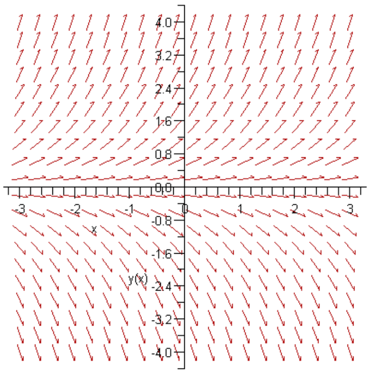
Direction field for y'=2xy



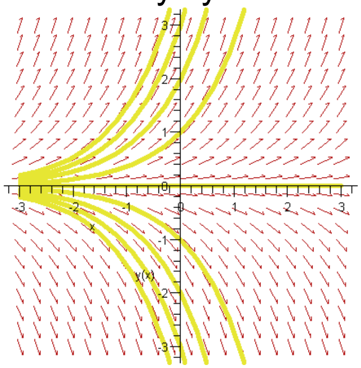
Direction field and some solutions of y'=2xy



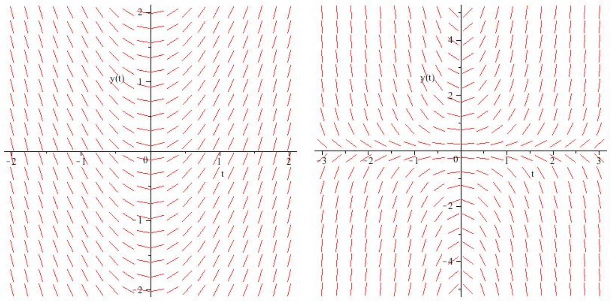
Direction field of y'=y

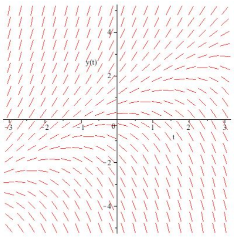


Direction field and some solutions of y'=y

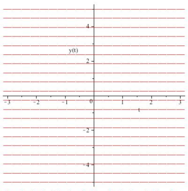


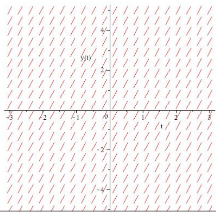
Direction fields for y'=2x and y'=2xy

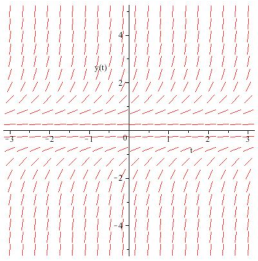




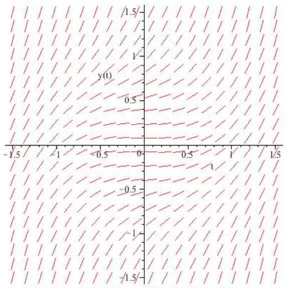
Direction fields. What are the differential equations?

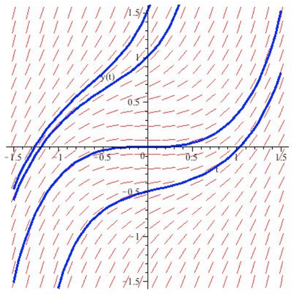






A direction field and some solutions.





Euler's method

Leonhard Euler 1707 - 1783

Euler method

Goal: Find an approximate numerical solution of y'=2x.y, y(1)=2 at a point close to 1.

First step

Recall: The tangent line is a good approximation to a curve on a small interval.

The equation of the line through (x_0,y_0) with slope m is y= y_0 +m(x-x_0)

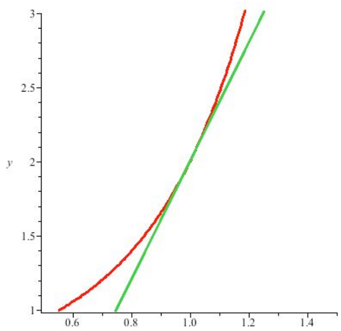
- We begin by approximating solution y = y(x) when x is close to 1.
- The solution (x, y(x)) passes through the point (1, 2).
- The value of y' at (1,2) is 2. x_0 . y_0 =2.1.2=4.
- The tangent line to the solution at the initial point (1,2) is y = 2+4(x-1)
- Thus if x_1 is close enough to 0, we can approximate y (x_1) by the line y_0+m(x_1-x_0)=2+4(x_1-1)

Euler method

Recall: The tangent line is a good approximation to a curve on a small interval.

Goal: Find an approximate solution of y'=2x.y, y(1)=2.

- Thus if x_1 is close enough to 0, we can approximate y (x_1) by the line y_0+m(x_1-x_0)=2+4 (x_1-1)



Euler's method

Given y'=F(x,y), y(x_0)=$y_{0,}$ h

$y_0 = y(x_0)$ initial condition

h stepsize

$x_i = x_0 + i.h$

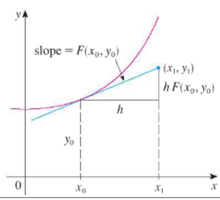
$y_1 = y_0 + hF(x_0, y_0)$

$y_2 = y_1 + hF(x_1, y_1)$

....

$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$

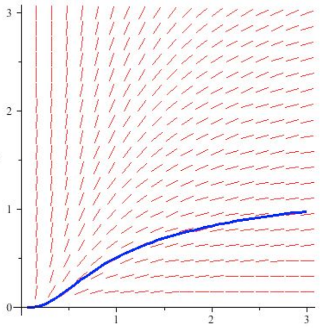
Given (x_i,y_i), and step size h, compute the "next" point (x_{i+1},y_{i+1}),



Euler's method

Given $y'=F(x,y)$, $y(x_0)=y_0$, h and n.

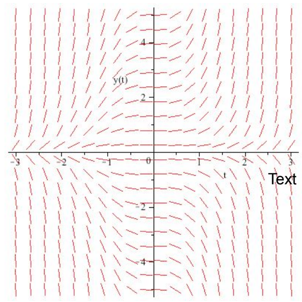
Estimate $y(x_n)$



Example: Use Euler's method to estimate y(2) where y(x) is the solution of the initial value problem $y'=y/x^2$, y(1)=0.5

1. with step size 1
2. with step size 0.5.
3. with step size 0.2

What is the step size given the best estimation?



Euler's method: Consider the equation $y'=xy^2$, y(0) =1.

Use Euler's method with step size 0.2 to estimate y(0.6)

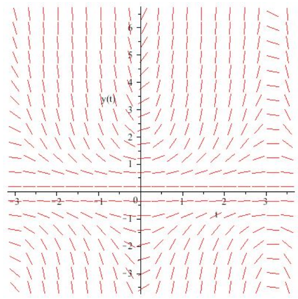
1, 1.04, 1.127

$$\frac{dy}{dx} = f(x)/g(y)$$

MAT132

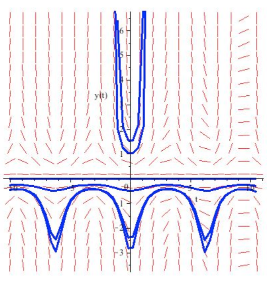
Separable equations

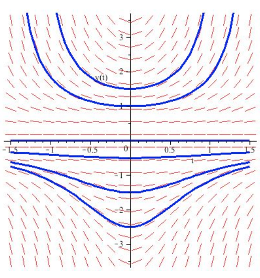
EXAMPLE: Find the solutions of $y'=\sin(x).y^2$



What are the equilibrium solutions of $y'=\sin(x).y^2$

Two different "windows" and some solutions of $y'=\sin(x).y^2$





Consider the equation $y'=\sin(x).y^2$

- write $f(x)=\sin(x)$, $g(y)= y^2$
- Then $dy/dx = f(x).g(y)$
- $\int(1/g(y))dy = \int f(x)dx$

A separable differential equation is a differential equation that can be written as $y'= f(x).g(y)$

Examples of separable differential equations

$y'=(x+e^x)\cos(y)$

$y'=\sin(x).y^2$

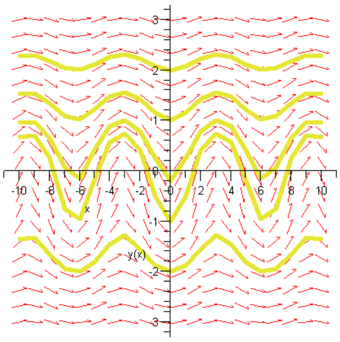
Examples of non separable differential equations

$y'=(x+e^y)\cos(x.y)$

$y'=\sin(x+y).y^2$

EXAMPLE: Find the solutions of $y' = \sin(x)/(1+\sin(y) + y^2)$

Direction field of $' = \sin(x)/(1+\sin(y) + y^2)$



Remark

- The solutions of the equation $y' = \sin(x)/(1+\sin(y) + y^2)$ are all the functions $y=y(x)$ which satisfy the equation $y - \cos(y) + y^3/3 = -\cos(x) + C$ but we it is not possible to give an explicit formula for y.

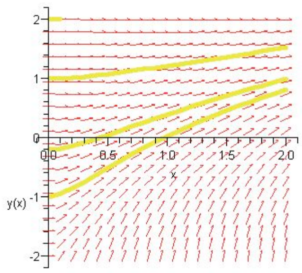
Example: Solve the differential equations

2 $\dfrac{dy}{dx} = \dfrac{\sqrt{x}}{e^y}$

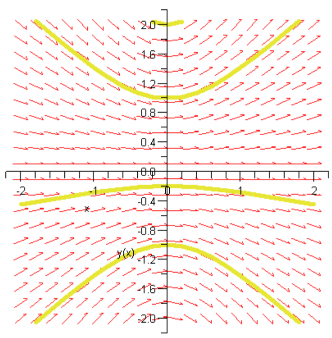
3. $(x^2+1)y' = xy$

6. $\dfrac{dy}{dt} = \dfrac{e^y \sin^2(t)}{y \sec(t)}$

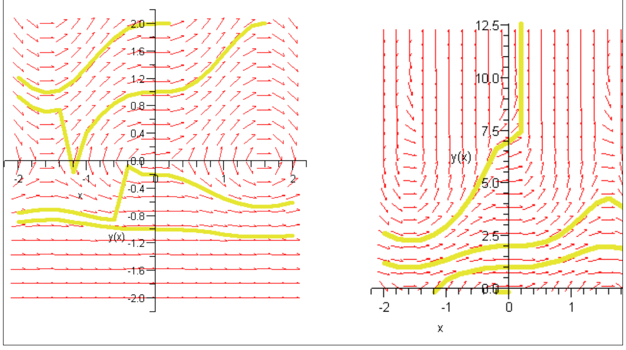
2. $\dfrac{dy}{dx} = \dfrac{\sqrt{x}}{e^y}$



$(x^2+1)y' = xy$



$$\frac{dy}{dt} = \frac{e^{y}\sin^{2}(t)}{y\sec(t)}$$



EXAMPLE: Find an equation of the curve that satisfies $dy/dx=4x^3y$ and whose y intercept is 7.