Problem. (35) Find the Taylor series of $f(x) = \sin x$ at $x = \pi$

(Solution) We begin by manually computing the first few derivatives of f evaluated at π to look for a pattern:

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\sin x$	0
1	$\cos x$	-1
2	$-\sin x$	0
3	$-\cos x$	1
4	$\sin x$	0

So we see that $f^{(n)}(\pi)$ is 0 for all even values of n and alternates between positive and negative 1 for odd values of n. So if $n = 2k+1$ then $f^{(n)}(\pi) = (-1)^{k+1}$. Notice that the exponent is $k+1$ since we need $f^{(n)}(\pi)$ to be -1 when $k = 0$ and hence $n = 1$. Therefore, our Taylor series is

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(x-\pi)^{2k+1}}{(2k+1)!}$$

Compare this to the McLaurin series for $\sin x$ which is

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

And notice that they are the same except for their centers and the exponent of -1. Think about what this means, and why it should be true (*Hint*: look at the graph of $\sin x$).

Problem. (36) Find the McLaurin series of $f(x) = \frac{x^3}{1+x}$

(Solution) This problem is difficult to do directly, but there is an easy way around: we can use the McLaurin series of $(1+x)^{-1}$, which is much easier to calculate, and then just multiply it by x^3 at the end. If $g(x) = (1+x)^{-1}$ we will leave it as an exercise to the reader to verify that:

$$g(x) = \sum_{k=0}^{\infty} (-1)^k x^k$$

Therefore, to obtain f we simply multiply g by x^3 which amounts to simply increasing the exponents by 3 and so we obtain:

$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^{k+3}$$

Problem. (37) Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ and (a) find the interval of convergence, (b) estimate $f(-1/2)$ by adding four terms, and (c) determine how many terms of the series $f(-1/2)$ are required to ensure that the sum is accurate to within 0.0001.

(Solution) (a) Let $a_n = \frac{x^n}{n^2}$ and apply the ratio test to determine the initial interval of convergence:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \frac{n^2}{x^n} \right| = \lim_{n\to\infty} \left| x \left(\frac{n}{n+1} \right)^2 \right| = \lim_{n\to\infty} \left| x \left(\frac{1}{1+\frac{1}{n}} \right)^2 \right| = |x|$$

And so if $|x| < 1$, the series will converge, which means that the interval of convergence is at least $(-1, 1)$. It now remains to check the endpoints:

Case 1. If $x = 1$ then $a_n = \frac{1}{n^2}$ which converges since it is a p-series with $p > 1$.

Case 2. If $x = -1$ then $a_n = \frac{(-1)^n}{n^2} = (-1)^n b_n$ where $b_n = n^{-2}$ is the nonalternating part of a_n. Since n^2 is increasing, its reciprocal b_n is decreasing. Furthermore, $\lim_{n\to\infty} b_n = 0$. Therefore, by the alternating series test, the series converges.

So the series converges at both endpoints and so the final interval of convergence is $[-1, 1]$.

(b) This is a direct computation. The first four terms give us the fourth partial sum

$$s_4(x) = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} = \frac{1}{144}\left(144x + 36x^2 + 16x^3 + 9x^4\right)$$

and so

$$s_4(-1/2) = -1031/2304 \approx -0.447483$$

(c) When $x = -1/2$ our series becomes

$$f(-1/2) = \sum_{n=1}^{\infty}(-1/2)^n/n^2 = \sum_{n=1}^{\infty}(-1)^n b_n$$

where $b_n = 2^{-n}n^{-2}$. We can now apply the Alternating Series Estimation Theorem which states that in a convergent alternating series $\sum(-1)^n b_n$ the error estimate of the first n terms, R_n, has absolute value at most b_{n+1}, i.e.

$$|R_n| \leq b_{n+1}$$

Therefore, to be accurate within 0.0001 we need

$$|R_n| = b_{n+1} = \frac{1}{2^n n^2} \leq 0.0001 = \frac{1}{10,000}$$

And so, by taking the reciprocal we need to determine when

$$10,000 \leq 2^n n^2$$

By manual calculation we find that the smallest value for which this is true is $n = 8$. So the approximation is sufficiently accurate when $n \geq 8$.

Problem. (38) Use series to evaluate the limit $\lim_{x\to 0} \frac{x^2 e^x}{\cos x - 1}$

(Solution) We will use the the McLaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Therefore

$$\lim_{x\to 0} \frac{x^2 e^x}{\cos x - 1} = \lim_{x\to 0} \frac{x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots}{-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots} = \lim_{x\to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{-\frac{1}{2!} + \frac{x^2}{4!} + \cdots} = -2$$