Second order differential equations

MAT 132

A ***differential equation*** is an equation in which the unknown is a function and where one or more of the derivatives of this function appears. In other words, it is an equation that relates a function with one or more of its derivatives.

The order of the highest order derivative of the unknown function is called the ***order*** of the differential equation.

Differential equation Example

$y'' + y = 0$

Some solutions of the example:

y=cos x, y=sin x and y=cos x - 3 sin x.

The family of solutions

y= A cos x + B sin x, where A and B are two constants.

A ***solution*** of a differential equation is any function that when substituted for the unknown function makes the equation an identity for all values of the variable in some interval.

The family of ***family of solutions of a differential equation*** the collection of all solutions of the differential equation.

A ***second order linear differential equation with constant coefficients*** is an equation is a differential equation of the form $A y'' + B y' + C y = f(x)$, where A, B and C are arbitrary constants and $A \neq 0$.

A special case is the differential equation $A y'' + B y' + C y = 0$, which is called ***homogeneous*** second order linear differential equation with constant coefficients.

- ***second order*** ″
- ***linear*** unknowns are only added (Compare to linear equation A x + B y =C)
- ***constant coefficients*** A,B , C are constants.
- ***homogeneous*** f(x)=0 for all x.

Example of a second order linear differential equation with constant coefficients $y''+y=0$

Some solutions of the example:

y=cos x, y=sin x and y=cos x - 3 sin x.

General solution (family of solutions)

$y= c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are two constants.

The ***general solution of a differential equation*** is the family of all solutions of the differential equation.

How do we find the general solution of a differential equation?

Two important theorems about the solution of second order linear differential equation with constant coefficients

If If y_1 and y_2 are both solutions of the equation

$$A y'' + B y' + C y = 0$$

and c_1 and c_2 are any two constants, then

$$c_1 y_1 + c_2 y_2$$

is also a solution of $A y'' + B y' + C y = 0$.

Example $y''+ y = 0$.

Two linearly independent solutions: y=cos x, y=sin x.

General solution (family of solutions)

$$y= c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 are two constants.

If If y_1 and y_2 are linearly independent (that is, one is not multiple of the other) solutions of the equation

$$A y'' + B y' + C y = 0$$

then all solutions can be written as

$$c_1 y_1 + c_2 y_2$$

for two constants c_1 and c_2.

More about the solution of second order linear differential equation with constant coefficients

1. $y'' - 5y' + 6y = 0$.
2. $y'' - 4y' + 4y = 0$.
3. $y'' - 4y = 0$.
4. $9y'' + 1 = 0$
5. $y'' + 3y' + 3y = 0$.

Find the roots of the characteristic equation in each of the above cases.

The equation

$$A x^2 + B x + C = 0$$

is the **characteristic equation** associated to the differential equation

$$A y'' + B y' + C y = 0$$

Case I: Characteristic equation with different (real) roots

Example $y'' - 5y' + 6y = 0$.
Find all solutions of the above equation, that can be written as

$$y = c_1 e^{rx},$$

where r is a real number.

If If y_1 and y_2 are linearly independent (that is, one is not a multiple of the other) solutions of the equation

$$A y'' + B y' + C y = 0$$

then all solutions can be written as

$$c_1 y_1 + c_2 y_2$$

for two constants c_1 and c_2.

If the roots r_1 and r_2 of the characteristic equation

$$A x^2 + B x + C = 0$$

are real and different the $e^{r_1 x}$ and $e^{r_2 x}$ are linearly independent solutions of the equation

$$A y'' + B y' + C y = 0.$$

If r_1 and r_2 are different (real) solutions of the characteristic equation $A x^2 + B x + C = 0$ then the general solution of $A y'' + B y' + C y = 0$ is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$, where c_1 and c_2 are two constants.

Case I: Characteristic equation with different (real) roots

1. $y'' - 5y' + 6y = 0$.
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4. $9y'' + 1 = 0$
5. $y'' + 3y' + 3y = 0$.

Find all solutions of each of the above equations.

The equation

$$A x^2 + B x + C = 0$$

is the **characteristic equation** associated to the differential equation

$$A y'' + B y' + C y = 0$$

If r_1 and r_2 are different (real) solutions of the characteristic equation

$$A x^2 + B x + C = 0$$

then the general solution of

$$A y'' + B y' + C y = 0$$

is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$, where c_1 and c_2 are two constants.

Case II: Characteristic equation with repeated roots

Example Find all solutions of the equation

$$y'' - 4y' + 4y = 0.$$

If If y_1 and y_2 are linearly independent (that is, one is not multiple of the other) solutions of the equation

$$A y'' + B y' + C y = 0$$

then all solutions can be written as

$$c_1 y_1 + c_2 y_2$$

for two constants c_1 and c_2.

If the characteristic equation

$$A x^2 + B x + C = 0$$

has only one real root r then e^{rx} and $x e^{rx}$ are linearly independent solutions of the equation

$$A y'' + B y' + C y = 0.$$

If the characteristic equation $A x^2 + B x + C = 0$ has only one real root then the general solution of $A y'' + B y' + C y = 0$ is is $y = c_1 e^{rx} + c_2 x e^{rx}$, where c_1 and c_2 are two constants.

Case III: Characteristic equation with complex roots

Example 1 Find all the solutions of the equation $y'' + 3y' + 3y = 0$.
Example 2 Find all solutions of the equation $9y'' + 1 = 0$.

If If y_1 and y_2 are linearly independent (that is, one is not multiple of the other) solutions of the equation

$$A y'' + B y' + C y = 0$$

then all solutions can be written as

$$c_1 y_1 + c_2 y_2$$

for two constants c_1 and c_2.

If the roots of the characteristic equation

$$A x^2 + B x + C = 0$$

are the complex numbers $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ then $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are linearly independent solutions of the equation

$$A y'' + B y' + C y = 0.$$

If the roots of the characteristic equation $A x^2 + B x + C = 0$ are the complex numbers $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ then the general solution of $A y'' + B y' + C y = 0$ is $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$ is where c_1 and c_2 are two constants.

Summary on solving the linear second order homogeneous differential equation

To find the general solution of the differential equation $A y'' + B y' + C y = 0$ we consider the characteristic equation:

$$A x^2 + B x + C = 0$$

Set $\Delta = B^2 - 4AC$.

	roots of the characteristic polynomial	General solution
$\Delta>0$	two distinct real roots r	c
$\Delta<0$	two complex roots $\alpha+i\beta$ and $\alpha - i\beta$	e sin(βx))
$\Delta=0$	one double real root r	c

Solving initial value problems

1. Solve the initial-value problem $y'' + 2y' + y = 0$, $y(0)=1$, $y(1)=3$.
2. $2y''+5y+3y=0$, $y(0)=3$, $y'(0)=-4$.