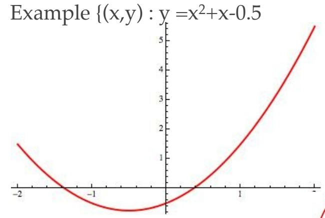
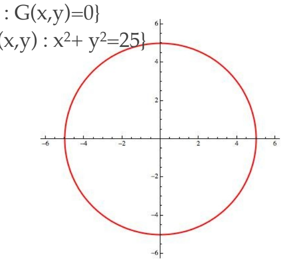
Areas and lengths in polar coordinates

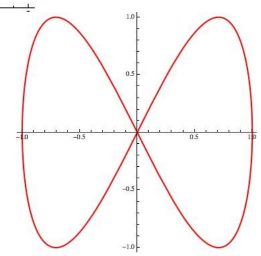
Different ways of representing curves on the plane

Given as a function
{(x,y), y =f(x)}
Example {(x,y) : y =x²+x-0.5



As the set of points satisfying an equation
{(x,y) : G(x,y)=0}
Example {(x,y) : x²+ y²=25}



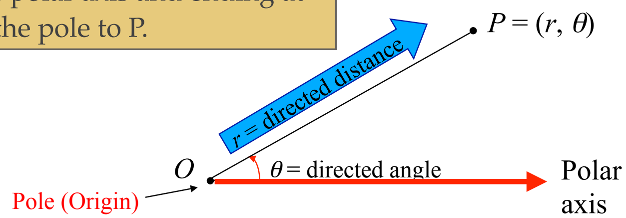


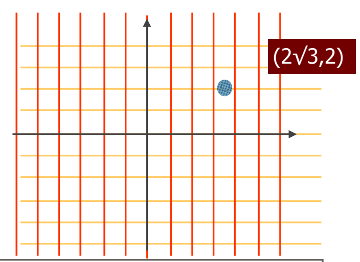
By parametric equation
x=f(t), y=g(t), t in [a,b]
Example x=sin(t), y=sin(2t),
t in [0,2π]

A different way of representing a point on the plane: polar coordinates

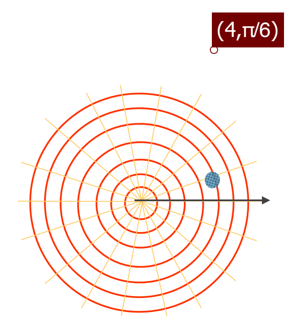
Fix
a point (the pole) and
a ray from the point (the polar axis)

A point P has polar coordinates (r, ϴ) if
1. The distance form the pole is |r|.
2. ϴ is the measure of the directed angle starting at the polar axis and ending at the ray from the pole to P.



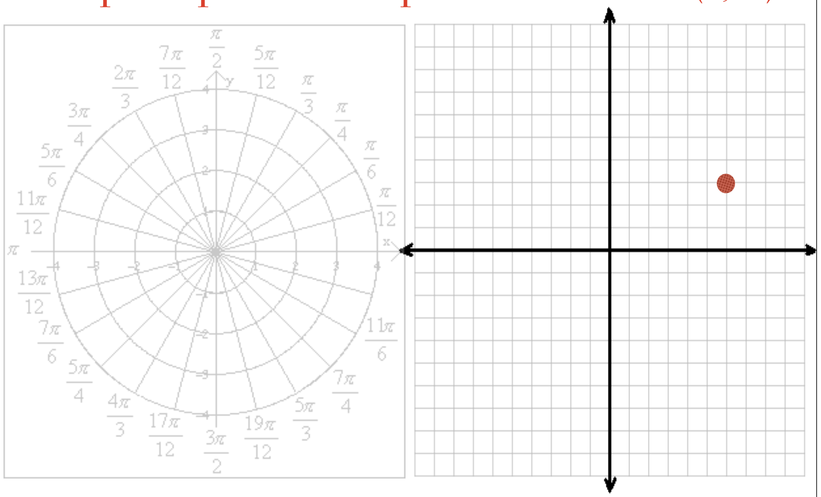


In **rectangular or cartesian coordinates**:
The plane is "organized" as grid of horizontal and vertical line lines.
A point is labeled with a pair of numbers that correspond to the vertical and the horizontal line the point belongs.



In **polar coordinates**:
We break up the plane with circles centered at the origin and with rays emanating from the origin.
We locate a point as the intersection of a circle and a ray.

How to plot a point P with polar coordinates (r,ϴ)



How to plot a point P with polar coordinates (r,ϴ)

- If r=0, P is the pole.
- If r>0, rotate the polar axis an angle ϴ (counterclockwise if ϴ>0, clockwise otherwise) and place P on this ray at distance r from the pole.
- If r<0, proceed as if r>0, but place P the point in the opposite ray, at distance -r from the pole.

Plot the points with polar coordinates given below.

- (0, 27)
- (4, π/6)
- (-3, π/2)

Points can be represented in more than one way in polar coordinates.

Find more ways to represent the points above.

$r = (x^2+y^2)^{1/2}$

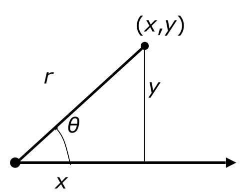
$\Theta = \arctan(y/x)$

If (x,y) in rectangular coordinates is given → The polar coordinates of the point are (r, ϴ)

$x = r.\cos\Theta$

$y = r.\sin\Theta$

If (r,ϴ) in polar coordinates is given → The rectangular coordinates of the point are (x,y)



Example

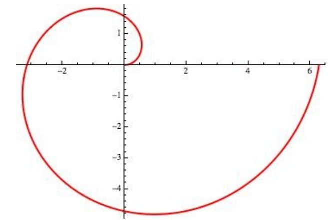
1. Find polar coordinates for a the point with rectangular coordinates (2√3,2)
2. Find the rectangular coordinates for the point with polar coordinates (-4, 5π/6)
3. Sketch a graph of the polar curve r = sinϴ - cosϴ
4. Sketch a graph of the polar curve r = cos(5ϴ)

Polar curves

A polar curve is a curve described by an a equation in polar coordinates.

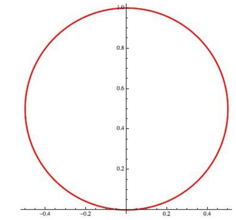
Plot the following examples

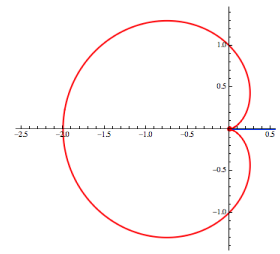
- r = 3 cos(2ϴ)
- r = (1-cosϴ)
- r=ϴ
- r = sin ϴ

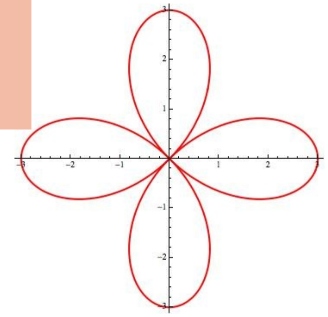


Match the curve with the equation

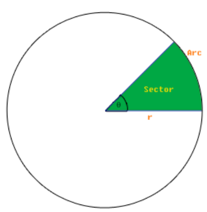
- r = 3 cos(2ϴ)
- r = (1-cosϴ)
- r=ϴ
- r = sin ϴ





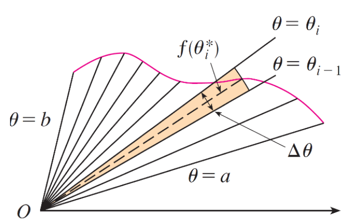


Area of a sector of a circle of radius r



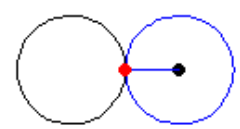
$$\frac{\theta}{2\pi} = \frac{A}{\pi r^2} \Rightarrow A = \frac{r^2\theta}{2}$$

Area A of a region bounded by a polar curve of equation r=f(ϴ), ϴ in [a,b]



$$\Delta\theta r^2/2 = \Delta\theta . f(\theta)^2/2$$

$$A = \frac{1}{2}\int_a^b f(\theta)^2\, d\theta$$



Consider the polar curve of equation
r = a (1-cosϴ)
(a is constant) ϴ in [0,2π]

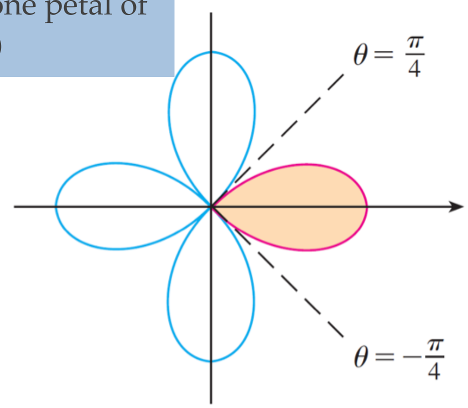
(This curve is called *cardiod*. The animation is from Wolfram http://mathworld.wolfram.com/Cardioid.html)

This curve has parametric equations
x=a cos(t)(1-cos(t))
y=sin(t)(1-cos(t))

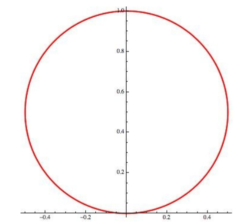
and its rectangular coordinates its points satisfy the equation
$(x^2+y^2+ax)^2=a^2(x^2+y^2)$

Example: Compute the area bounded by the cardiod
r = (1-cosϴ)
ϴ in [0,2π]

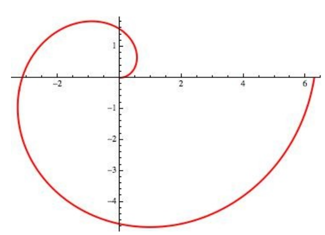
- Find the area inside one petal of the curve r=3cos(2 Θ)



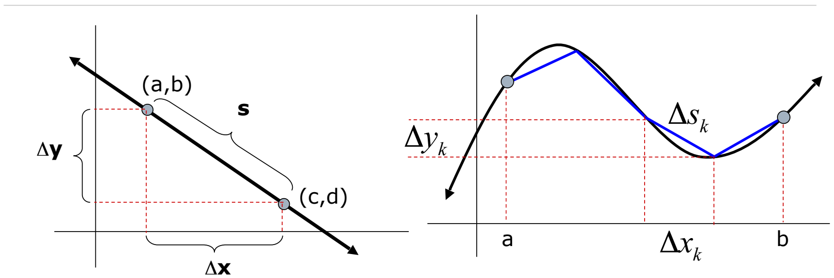
- Find the area enclosed by the curve r=sin(Θ)



- Find the area enclosed by the curve r=Θ , Θ in [0,2π] and the positive x-axis



Arc length review



$$\int_a^b \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2}\,dt$$

Length of a curve in polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \& \quad r = f(\theta)$$

Given the polar curve

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Differentiate with respect to Θ

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Square and add

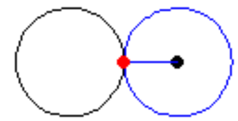
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Using the formula for parametric curves

$$\int_a^b \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2}\,dt$$

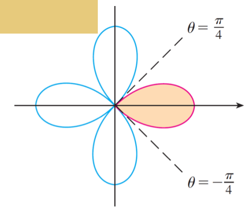
$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

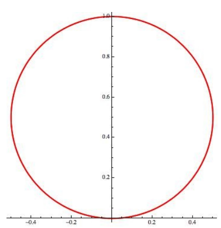


Example: Express (but not evaluate) as an integral the arc length of the cardiod lying above the x-axis.
r = 1-cosΘ
Θ in [0,2π]

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

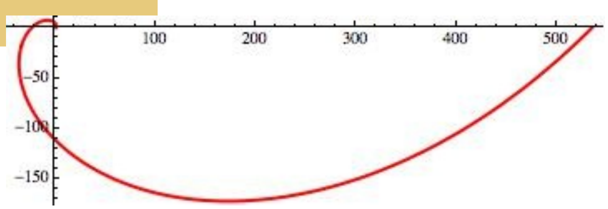
Example: Express (but not evaluate) as an integral the length of a petal of 3 cos(2Θ)



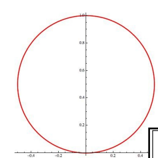


Example: Compute the arc length of the circle r = sinΘ
Θ in [0,2π]

Example: Compute the length arc of the spiral r = e^Θ
in [0,2π]



Example: Compute the arc length of the circle r = sinΘ
Θ in [0,2π]

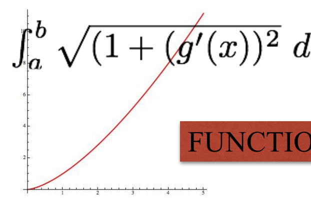


POLAR

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Example: Compute the arc length the graph of the curve $y=x^{3/2}$,
x in [0,5]

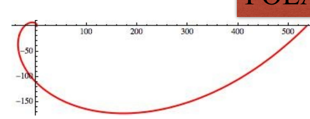
$$\int_a^b \sqrt{(1 + (g'(x))^2}\;dx$$



FUNCTION

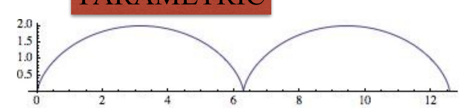
Example: Compute the length arc of the spiral r = e^Θ
in [0,2π]

POLAR



Example: Compute the length two arcs of the cycloid x= Θ - sinΘ, y=1-cos Θ. (Hint (1-cos(t))=2 sin²(t/2))

PARAMETRIC



$$\int_c^d \sqrt{((f'(t))^2 + (g'(t))^2}\;dt$$