MAT 132 Series

8.4 Other convergency tests
Alternating series
Absolutely convergence series
Ratio Test

A series
$$\sum_{i=1}^{\infty}(-1)^n a_n$$
is called ***alternating*** if $\{ a_n \}$ is a sequence of positive terms, that is $a_n > 0$ for all n.

TWO EXAMPLES OF ALTERNATING SERIES

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n(n+2)}{n(n+1)}$$

If $\{ a_n \}$ is a sequence such that

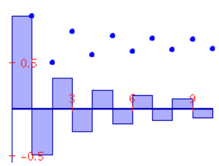
$a_n > 0$,

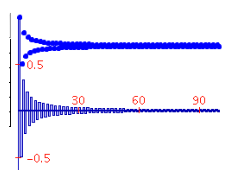
$a_n \geq a_{n+1}$, and

$a_n \to 0$ when $n \to \infty$

then the series

CONVERGES





Study the convergence of the the series.
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$
$$\sum_{n=1}^{\infty}\frac{(-1)^n(n+2)}{n(n+1)}$$

In order to check whether a sequence is decreasing, one can
1. compare consecutive terms or
2. if the sequence is defined by a function, check derivative of the functionis negative.

The 15th partial sum of the series
$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$$
is -0.824542. In symbols

$$\sum_{n=1}^{15}(-1)^n\frac{1}{n^2} = -0.824542$$

Estimate the error in using -0.824542 to approximate the total sum of the series.

Alternating Series Estimation Theorem If $s = \Sigma(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

$$\text{(i)}\ b_{n+1} \leqslant b_n \quad \text{and} \quad \text{(ii)}\ \lim_{n\to\infty} b_n = 0$$

then $$|R_n| = |s - s_n| \leqslant b_{n+1}$$

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{3} n\pi\right)}{n^2+2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+2)}{n(n+1)}$$

Definition A series Σa_n is called **absolutely convergent** if the series of absolute values $\Sigma |a_n|$ is convergent.

1 Theorem If a series Σa_n is absolutely convergent, then it is convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n+2)}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!}$$

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n$$

The Ratio Test

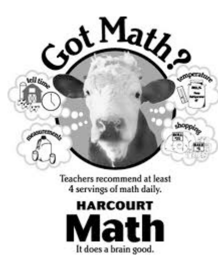
(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of Σa_n.

MAT 132

8.5 Power Series



A **power series in x** is a series that can be expressed as

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \cdots$$

The coefficients $c_0, c_1, c_2 \ldots$ are constants.

We also use the notation

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \cdots$$

By replacing "x" by a real number, we obtain a power series

Examples

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n\, 3^n} \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

A power series in (x-a) is a series that can be expressed as

$$\sum_{n=0}^{\infty} c_n(x-a)x^n = c_0 + c_1(x-a)x + c_2(x-a)x^2 + \cdots + c_n(x-a)^n \cdots$$

The coefficients $c_0, c_1, c_2\ldots$ are constants.

"a" is also a constant.

Examples

$$\sum_{n=0}^{\infty} 3\,(x-1)^n \qquad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (x+2)^n}{n\,3^n}$$

Examples

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n\,3^n} \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sum_{n=0}^{\infty} 3\,(x-1)^n$$

what is the key question about infinite series?

Does it converge?

- If we replace "x" by a number, a powers series becomes an infinite series. Thus, in the case of power series we ask:

For what values of "x" does the power series converge?

Find the values of x for which each of the the power series below is convergent.

$$\sum_{n=0}^{\infty} 3\,x^n \qquad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (x+2)^n}{n\,3^n} \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sum_{n=0}^{\infty} n!\,x^n$$

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Which of the possibilities of the theorem above hold for each of the power series?

$$\sum_{n=0}^{\infty} 3\,x^n \qquad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n\,3^n} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (x+2)^n}{n\,3^n}$$

$$\sum_{n=0}^{\infty} 3\,(x-1)^n \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sum_{n=0}^{\infty} n!\,x^n$$