MAT 132 Series

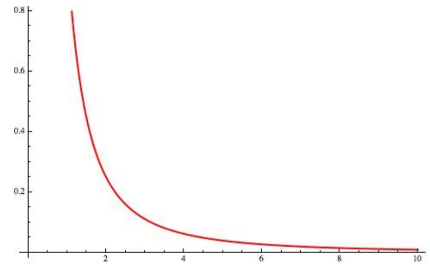
8.3 Some convergency tests and estimating sums

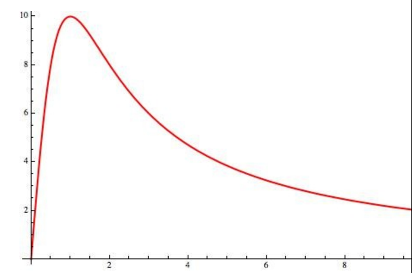
Consider functions $f(x) = 20x/(x^2+1)$ and $g(x) = 1/x^2$

Note that both the functions are positive and decreasing for $x \geq 1$

$f(x)=20x/(x^2+1)$

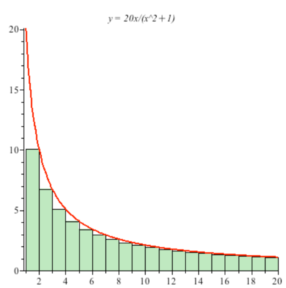
$g(x)=1/x^2$

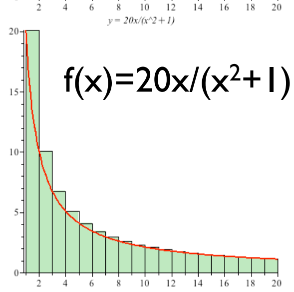




Consider the sequences $\{a_n\}$ and $\{b_n\}$ defined respectively by functions $f(x) = 20x/(x^2+1)$ and $g(x) = 1/x^2$

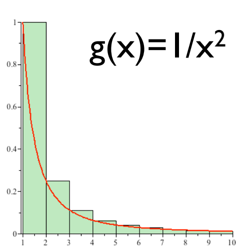
We want to study whether the corresponding series are convergent.

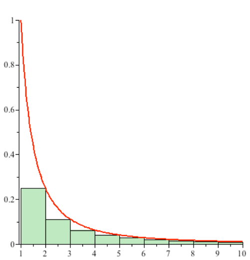




$f(x)=20x/(x^2+1)$

$g(x)=1/x^2$



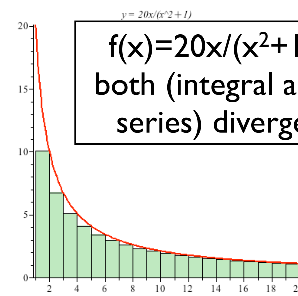


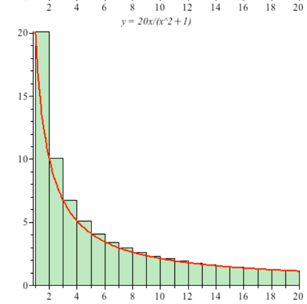
Integral test theorem: If $\{a_k\}$ is a sequence defined by a function f, that is, $a_k = f(k)$ ($k = 1, 2, 3,\ldots$) and the function f is positive, continuous, decreasing for $x \geq 1$

Then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$

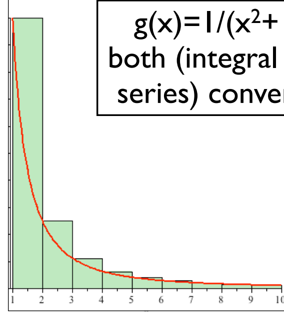
both converge or both diverge

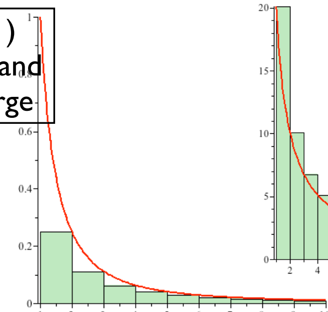
$f(x)=20x/(x^2+1)$ both (integral and series) diverge





$g(x)=1/(x^2+1)$ both (integral and series) converge





Integral test theorem: If $\{a_k\}$ is a sequence defined by a function f, that is, $a_k = f(k)$ (k = 1, 2, 3,…) and the function *f* is positive, continuous, decreasing for $x \geq 1$

Then $\sum_{k=1}^{\infty} a_k$ *and* $\int_1^{\infty} f(x)\,dx$

both converge or both diverge

Important Example

Use the integral test to determine wether the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

is convergent or divergent.

1 The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Examples to remember:

- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if p>1.
- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if p≤1.
- The geometric series $\sum_{n=1}^{\infty} a r^n$ is convergent if -1<r<1.
- The geometric series $\sum_{n=1}^{\infty} a r^n$ is divergent if r≤-1 or r≥1.
- The arithmetic series $\sum_{n=1}^{\infty} (a + (n-1)d)$ is divergent if d≠0 or a≠0.

Example Determine whether the series below are convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$$

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)\,(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + \pi}$$

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{3} n\pi\right)}{n^2 + 2}$$

In order to apply the comparison test, first find the term of a series you know that is "similar" to the term of the series you want to study.

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(b) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Example Determine whether the series below are convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+4}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{n^2+2}{\sqrt{4n^4+10}}$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Series Summary: Tests and Important Examples

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\sum_{n=1}^{\infty} a\,r^n$$

- Integral test
- Comparison test
- Limit comparison test
- Divergence test: If the sequence does not converges to 0, then the corresponding series diveverges.

Definition of the remainder after k terms.

- If an infinite series is convergent and the sum is S, then the remainder after k terms, obtained by approximating the sum of the series by the k-th partial sum s_k, is denoted by R_k and is $R_k = S - s_k$.

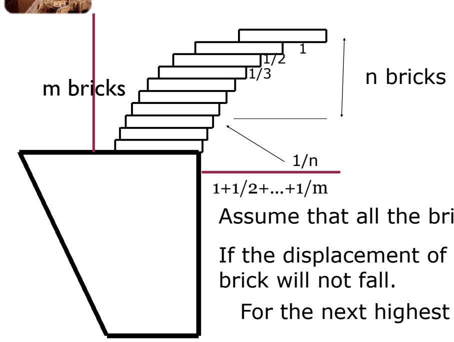
- Example: Find R_3 for the geometric series $\sum_{n=1}^{\infty} \frac{4}{5^n}$
  - If we approximate the series by s_3, what is the error?

Example

- Approximate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the first 4 terms.
- Estimate the error of the approximation.
- How many terms are required to ensure that the sum is accurate to within 0.0001?

3 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$



To build a cupola the builder lays bricks above each other and displaces them as far to the right as possible without making the building collapse. This is indicated in the picture on the left.

Assume that all the bricks have the length 2 units.

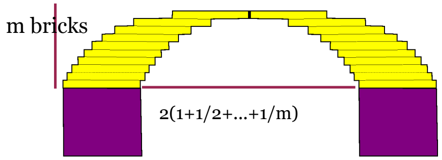
If the displacement of the highest brick is 1 unit, the brick will not fall.

For the next highest brick the displacement can be ½.

The displacement of the n^{th} highest brick can be 1/n.

Since the harmonic series diverges, one can build arbitrarily wide cupola in this way.

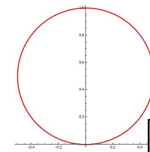
Cathedral of Florence and the Cupola of Brunelleschi



Since the harmonic series diverges, one could build arbitrarily large cupola whose stability is based on this fact.

Prime example of a large cupola is the cupola of the cathedral of Florence. This cupola was designed by Brunelleschi. The cupola was completed in 1436. The shape and the width of the cupola of Brunelleschi suggest, however, that its stability is not based on the divergence of the harmonic series.

Example: Compute the arc length of the circle r = sinΘ
Θ in [0,2π]

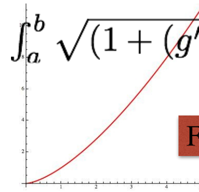


POLAR

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example: Compute the arc length the graph of the curve $y=x^{3/2}$, x in [0,5]

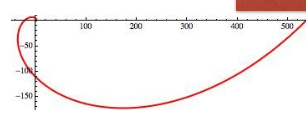
$$\int_a^b \sqrt{(1 + (g'(x))^2}\ dx$$



FUNCTION

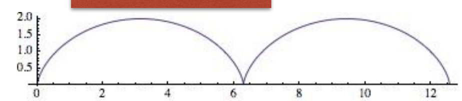
Example: Compute the length arc of the spiral $r = e^{\Theta}$ in [0,2π]

POLAR



Example: Compute the length two arcs of the cycloid x= Θ - sinΘ, y=1-cos Θ. (Hint (1-cos(t))=2 sin²(t/2))

PARAMETRIC



$$\int_c^d \sqrt{((f'(t))^2 + (g'(t))^2}\ dt$$

(6) Determine whether the each of the series is convergent or divergent. If it is convergent, find its sum. (Justify your answers)

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

(b) $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+3)} + \frac{1}{(-10)^n}\right)$

(c) $\sum_{n=1}^{\infty} \frac{n^2-4}{2n^2+3}$

(d) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

(7) Represent the number $3.4\overline{15} = 3.4151515\ldots$ as quotient of integers.

(8) (a) Approximate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the first 4 terms.

(b) Estimate the error of the approximation.

(c) Determine how many terms are required to ensure that the sum is accurate to within 0.0001.

(1) The region bounded by the curves $y = x^2$ and $x = y^2$ is rotated about the x-axis. Set up an integral for the volume of the resulting solid by two different methods.

(2) Set up (but do not evaluate) an integral for the length of the curve $y = x^{3/2}$, for $x \in [0, 4]$.

(3) Set up integral for the length of and the area of the inside loop of the polar curve $r = 1 - 2\cos(\theta)$.

(4) A tank on the shape of a sphere of radius $10ft$ is full of oil weighting $50\text{lb}/\text{ft}^3$. How much work is done by pumping the oil through a hole in the top?

(5) If 6 J of work is needed to strectch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12cm to 14cm, what is the natural length of the spring? (Recall Hooke's law: the force required to mantain a spring stretched x units beyond its natural length is proportional to x).

(6) Determine whether the each of the series is convergent or divergent. If it is convergent, find its sum. (Justify your answers)