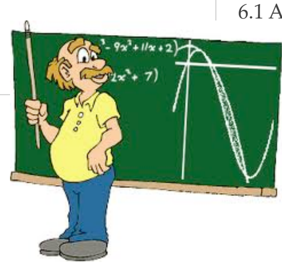
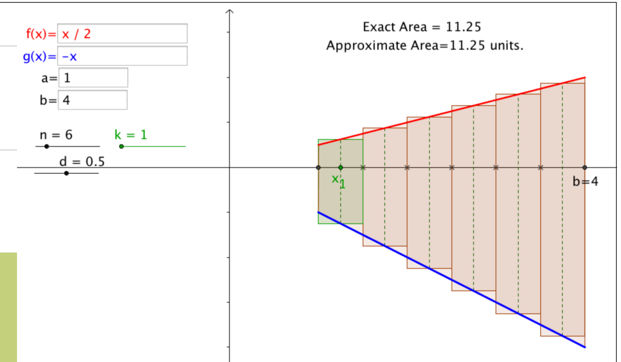
MAT 132

6.1 Areas between curves



Find the area between the curves $f(x)= x/2$ and $g(x)= -x$ on the interval [1,4]



Theorem

- Consider two continuous functions f(x) and g(x) both defined on the interval [a,b].
- Suppose that $f(x)\geq g(x)$ for all x in [a,b]
- Then the area A of the region bounded by the curves $y=f(x)$ and $y=g(x)$ and the lines $x=a$ and $x=b$ is

$$A=\int_a^b f(x)-g(x)dx$$

Find the area of the region bounded by the curves in the following cases

1. $y=x^2$ and $y=-x^2+4$. $\dfrac{16\sqrt{2}}{3}$

2. $y^2=2x-2$ and $y=x-5$. $-\dfrac{8\sqrt{10}}{3}$

Recipe to find the area bounded by curves.
Find intersection points. In most cases, these points will determine the limits of integration .
Sketch a figure.
Compute the definite integral.

Sometimes you will need to "rotate" the figure $\pi/2$ (considering x as a function of y)

<u>Area enclosed by parametric curves</u>

Theorem: The are the region bounded by the curve $x=f(t)$, $y=g(t)$, where t in [α,ß] and the x-axis is

$$\int_\alpha^\beta g(t)f'(t)dt$$

Examples:

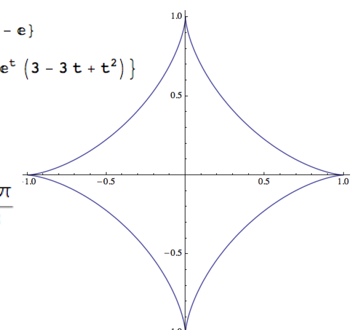
1.Find the area enclosed by the x-axis and the curve given by parametric equations $x=1+e^t$ and $y=t-t^2$.

$\{3-e\}$

$\{-e^t(3-3t+t^2)\}$

3.Find the area of the asteroid of equation $x=\cos^3(t)$, $y=\sin^3(t)$, t in $[0,2\pi]$.

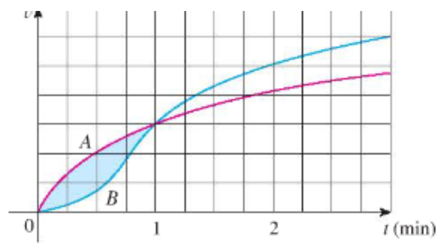
$\dfrac{3\pi}{8}$



4.Find the area of the ellipse of equation $(x/a)^2+(y/b)^2=1$ using parametric equations.

$a\,b\,\pi$

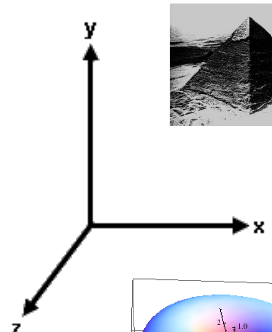
26. Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.
(a) Which car is ahead after one minute? Explain.
(b) What is the meaning of the area of the shaded region?
(c) Which car is ahead after two minutes? Explain.
(d) Estimate the time at which the cars are again side by side.

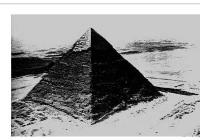


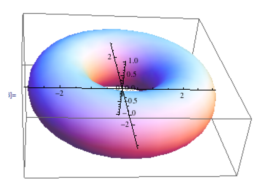
29. If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \leqslant t \leqslant 10$. What does this area represent?

Volumes

- R^3.
- Volumes of solids
- Solids of Revolution (a curve rotates about a line.)
- Volumes of solids of revolution
  - The disk method.
  - The washer method
  - Cylindrical shell (next class)







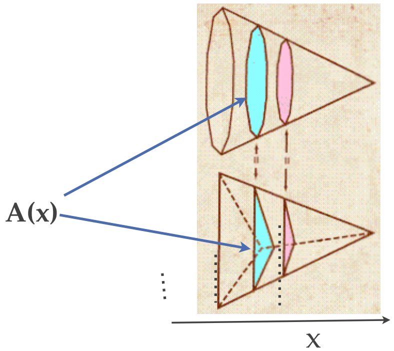
Volumes

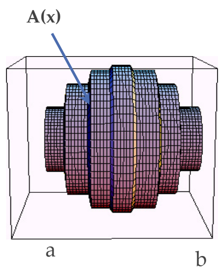
- To estimate the volume of the loaf of bread, we slice it, find the volume of each slice and add up all those volumes.
- The volume of each slice is approximately, the area of the slice multiplied by the height (thickness).

What can be do to get a better estimation?

Denote the cross-sectional area of the solid in the plane perpendicular to the x-axis by A(x).
If A is a continuous function, then the volume of the solid that lies between x=a and x=b is

$$\int_a^b A(x)\,dx$$

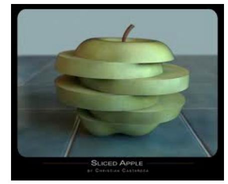




Make sure you understand in which direction you slice. When you use the formula

$$\int_a^b A(x)\,dx$$

this direction is perpendicular to "x"



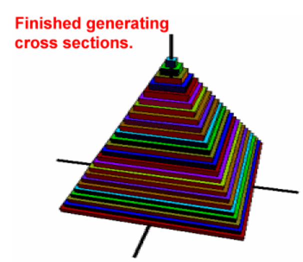


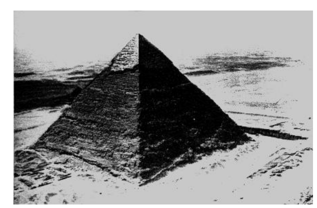

Computing the volume of a solid

1. Decompose the solid into small parts, each of which has a volume that can be approximated by an expression of the form $f(x_k)\Delta x_k$. Then the total volume can be approximated by th expression $\sum_{k=0}^{n} f(x_k)\Delta x_k$
2. Show that the approximation becomes better and better when n goes to infinite and each Δx_k approaches 0. Thus $Volume = \lim_{n\to\infty} \sum_{k=0}^{n} f(x_k)\Delta x_k$
3. Express the above limit as a define integral $\int_a^b f(x)dx$
4. Evaluate the integral to determine the volume

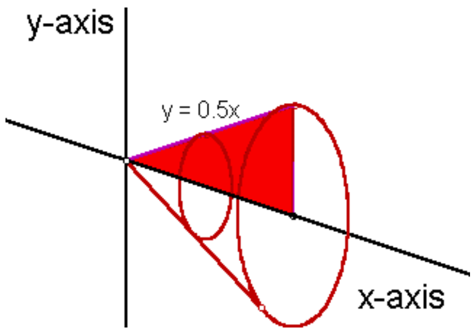
A right pyramid 4 ft. high has a square base measuring 1 ft. on a side.

Find its volume.

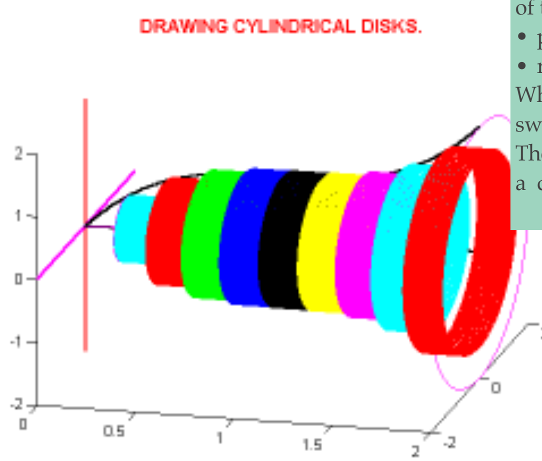




Find the volume of the cone obtained by rotating about the x-axis the segment of the line y=0.5x between 0 and 1.



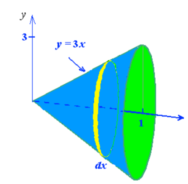
Solid of revolution - The disk method



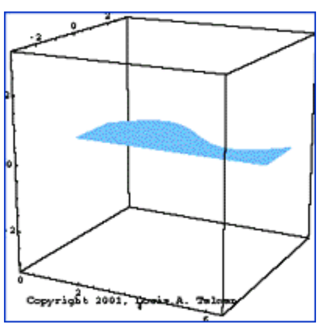
A typical element of area (a “cut” of the solid),
- perpendicular to the x-axis
- radius |f(x)|

When revolved about the x-axis, it sweeps out a circle of area π (f(x)).
The “thickened” element of area is a disk, with volume π (f(x)) dx

Example



Solid of revolution - The disk method



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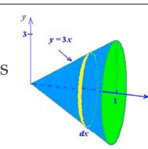
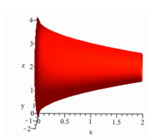
The curve depicted is the graph of the function $f(x)=1.3 + 2.3x - 2x^2 + 0.4x^3$, x in [0,3]. Express the volume of the solid obtained by rotating the curve about the x-axis as a definite integral.

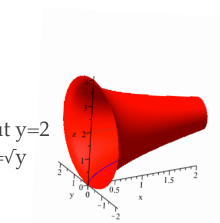
Rotating a curve about different lines

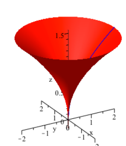
y=3x
about x axis



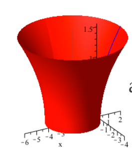


about y=2
x=√y





about y axis
$x=y^2$

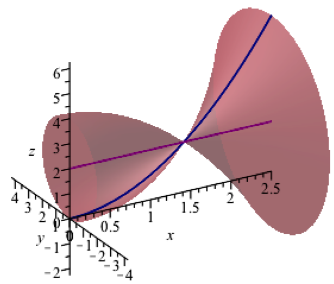


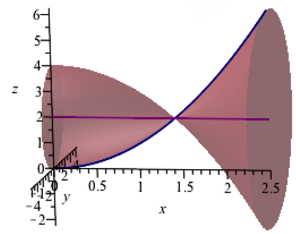
about x= - 3
$x=y^2$

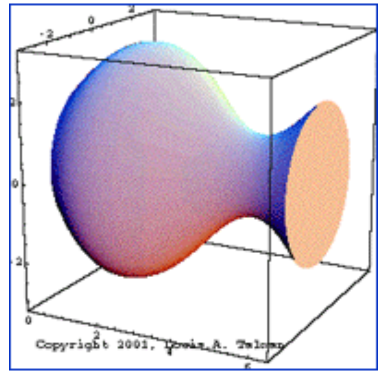
Compute the volume of a solid of revolution

A solid of revolution is formed when the region bounded by the curves $y=x^2$, x=2.5 and the -axis is rotated about the line y=2 . Find the volume the method of disks

$$\pi \left(\int_a^b Radius(x)^2 \, dx \right)$$

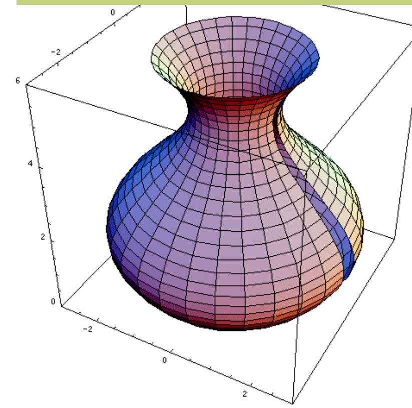






Rotating a curve about the x-axis

Rotating a curve about the y-axis



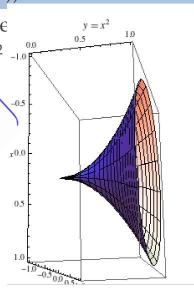
Solid of revolution - The disk method

Rotating a curve about the x-axis
A typical ***element of volume*** is a disk obtained by revolving about the x-axis a thin rectangle perpendicular to the x-axis of height $|f(x)|$
When this rectangle is rotated about the x-axis, it sweeps out a circular disk of volume $\pi (f(x))^2 dx$.

Rotating a curve about the y-axis
A typical ***element of volume*** is a disk obtained by revolving about the y-axis a thin rectangle perpendicular to the y-axis of height $|g(y)|$
When this rectangle is rotated about the y-axis, it sweeps out a circular disk of volume $\pi (g(y))^2 dy$.

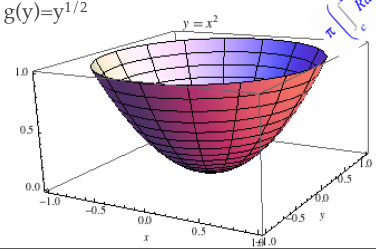
Example
$f(x)=x^2$

$$\pi \left(\int_a^b Radius(x)^2 \, dx \right)$$

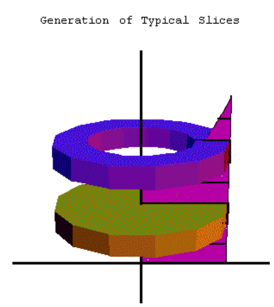


Example:
$g(y)=y^{1/2}$

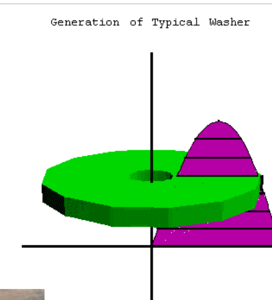
$$\pi \left(\int_c^d Radius(y)^2 \, dy \right)$$



Washers Method

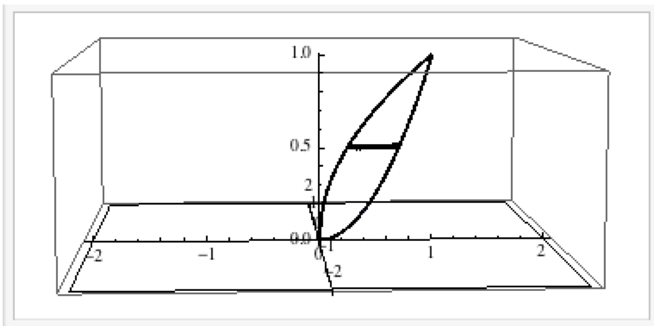


Find the volume of the solid of revolution formed by rotating the region bounded by the x-axis, the curve x=2 and the curve $y=x^2+2$ about the y-axis.

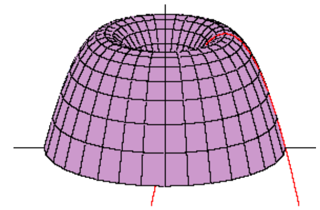


Find the volume of the solid of revolution formed by rotating the region bounded by the x-axis and the curve y=x(x-2) about the y-axis.

Find the volume of the solid generated by rotating the region of the *xy*-plane bounded by the curves $y=x^2$ and $y=x^{1/2}$ about the y-axis.

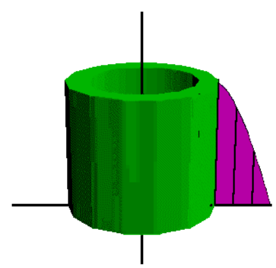


The solid of revolution



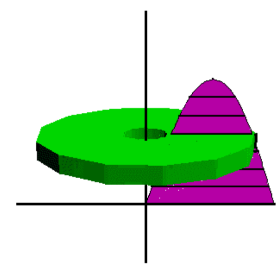
Cylindrical Shell

Generation of Typical Shell

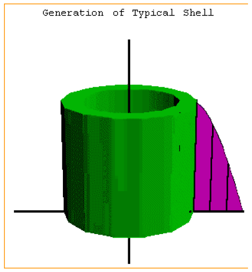


Washer

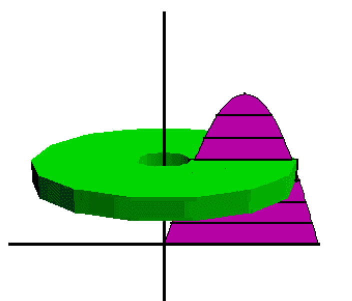
Generation of Typical Washer



Generation of Typical Shell



Generation of Typical Washer



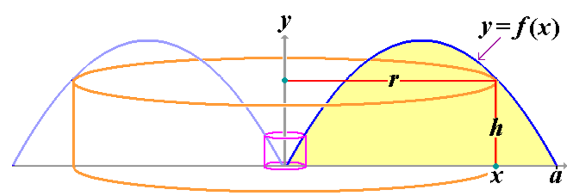
Shell method
A typical element of volume is a , cylindrical shell of volume 2π x(f(x))dx.

Washer Method
A typical element of volume is a a circular disk of volume π((f(y))^2- (g(y))^2 dy.

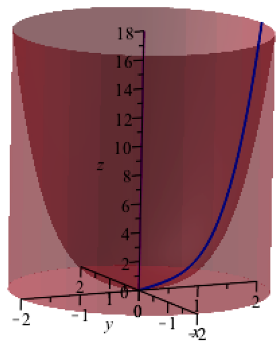
$$V = \int_a^b 2\pi x f(x)\,dx$$

$$\pi \int_a^b (g(y)^2 - f(y)^2)dy$$

Cylindrical shells

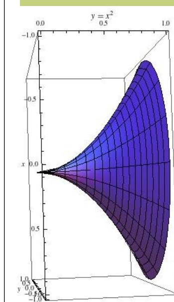


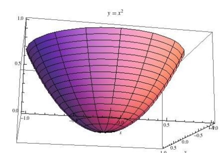
1. The region bounded by the curves $y=x^4+x$, the x-axis and the line $x=2$ is revolved about the y-axis. Find the volume of the obtained solid. ($80\pi/3$)
2. The region bounded by the curves $y=x^4+x$, the y-axis and the line $x=2$ is revolved about the y-axis. Find the volume of the obtained solid.
3. Observe that by adding 2. and 3 you obtain 72π, the volume of the cylinder of height 18 and radius 2.

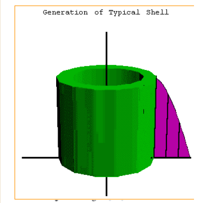


Summary

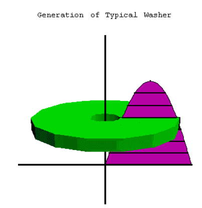
The solids below are obtained by rotating a region of the plane about vertical or horizontal axes. They are called *solids of revolution*.







$V = \int_a^b 2\pi x f(x)\, dx$

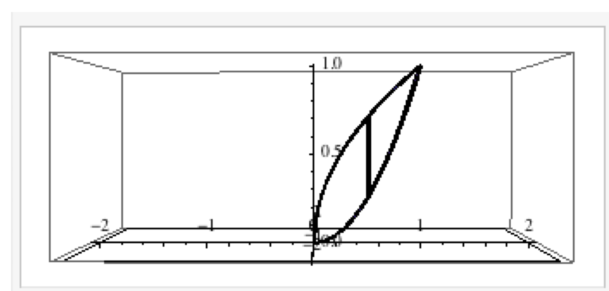


$\pi \int_a^b (g(y)^2 - f(y)^2)dy$

We discussed how to find the volume of solids of revolution by the washer method or the cylindrical shell method.

Example

- The region bounded by the curve $y=4-x^2$, the x axis and the line $x=2$ is rotated about the x-axis. Find the volume of the solid generated using the disk method and the shell method. Both methods should give the same answer!



13–18 The region enclosed by the given curves is rotated about the specified line. Find the volume of the resulting solid.

13. $y = 1/x,\ x = 1,\ x = 2,\ y = 0;$ about the x-axis

14. $x = 2y - y^2,\ x = 0;$ about the y-axis

15. $x - y = 1,\ y = x^2 - 4x + 3;$ about $y = 3$

16. $x = y^2,\ x = 1;$ about $x = 1$

17. $y = x^3,\ y = \sqrt{x};$ about $x = 1$

18. $y = x^3,\ y = \sqrt{x};$ about $y = 1$