MAT 132

Section 5.10 Improper integrals

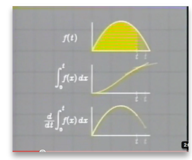
The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

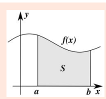
The Fundamental Theorem of Calculus is indeed, FUNDAMENTAL.

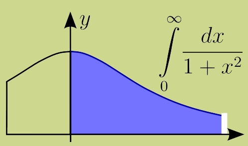
It gives a "recipe" to compute definite integrals.

Otherwise, we would have to spent our lives calculating Riemann sums, rectangle by rectangle.

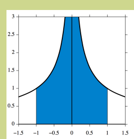


We have been working on definite integrals of **continuous functions** over **closed** intervals.





$$\int_0^{\infty} \frac{dx}{1+x^2}$$



In certain cases, it is possible to find "areas under the curve" ("almost but not quite" definite integrals) of

- functions that have a point of discontinuity or
- integrals defined over infinite intervals

These are called improper integrals.

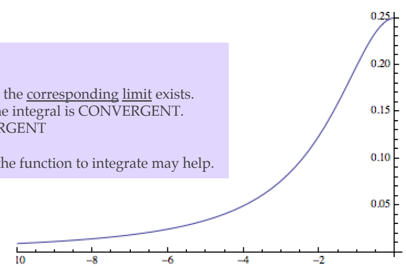
Infinite intervals

Determine whether the integral is convergent or divergent. Evaluate if it is convergent.

$$\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx$$

Strategy:
1. Find antiderivative
2. Determine whether the corresponding limit exists.
3. If the limit exists, the integral is CONVERGENT.
Otherwise, it is DIVERGENT

Sketching a graph of the function to integrate may help.



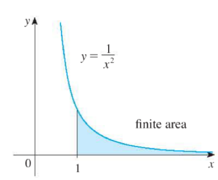
Infinite intervals

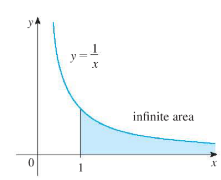
Determine whether the integral is convergent or divergent. Evaluate if it is convergent.

$$\int_3^{\infty} \frac{1}{x}\,dx$$

Strategy:
1. Find antiderivative
2. Determine whether the corresponding limit exists.
3. If the limit exists, the integral is CONVERGENT.
Otherwise, it is DIVERGENT

Sketching a graph of the function to integrate may help.



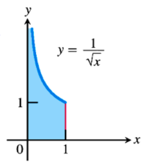


For which values of p is the integral $\int x^{-p}\,dx$ convergent?

Discontinuous integrands

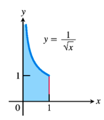
$$\int_a^b f(x)\,dx$$

1. f is continuous in [a,b) and discontinuous at b.
2. f is continuous in (a,b] and discontinuous at a.
3. f is continuous in [a,c) and (c,b] for some c in [a,b]



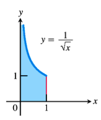
Strategy:
1.Find antiderivative.
2. Compute the appropriate one-sided limit (or one-sided limits in case 3).
3. If the limit (or limits) exists, then the integral is CONVERGENT. Otherwise, it is DIVERGENT.
4. If it s convergent, the integral is the limit (or sum of limits) you computed.
Sketching a graph of the function to integrate may help.

Example: Evaluate the integral, if it is convergent.



$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \qquad \int_0^1 x\ln(x)\,dx$$

Example: Evaluate the integral, if it is convergent.



$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \qquad \int_0^1 x\ln(x)\,dx$$

$$: -\frac{x^2}{4} + \frac{1}{2} x^2 \,\text{Log}[x]$$

$$: -\frac{1}{4}$$

A comparison test

If f and g are continuous functions with $0 \le f(x) \le g(x)$
For all $x \ge a$. Then.....

$\int_a^\infty f(x)dx$ is convergent if $\int_a^\infty g(x)dx$ Converges

A the integral of a function f converges if its values are smaller than another function known to converge.

$\int_a^\infty g(x)dx$ is divergent if $\int_a^\infty f(x)dx$ Diverges

A function diverges if its values are larger than another function known to diverge.

Use a comparison test to determine whether the integrals are convergent or divergent.

$$\int_2^\infty \frac{\sin(x)^2}{x^2}\,dx \qquad \int_2^\infty \frac{x+1}{\sqrt{x^4 - x}}\,dx$$

Infinite intervals

Determine whether the integral is convergent or divergent. Evaluate if it is convergent.

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx$$

$$\int_{-\infty}^\infty \frac{1}{e^x + e^{-x}}\,dx$$

Strategy:
1. Find antiderivative
2. Determine whether the corresponding limit exists.
3. If the limit exists, the integral is CONVERGENT. Otherwise, it is DIVERGENT

Sketching a graph of the function to integrate may help.

$$\int_0^\infty \sin(x)\,dx$$

Solutions

$\int_{1}^{\infty} \frac{\ln(x)}{x^2}\,dx$		$-\frac{\ln(x)}{x} - \frac{1}{x}$	1
$\int_{3}^{\infty} \frac{1}{x}\,dx$			
$\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx$	$\frac{1}{4}\pi$	$\frac{1}{2}\arctan\left(\frac{1}{2}x\right)$	
$\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx$	$\frac{1}{2}\pi$	$\arctan(e^x)$	