An infinite crowd of mathematicians enters a bar. The first one orders a pint, the second one a half pint, the third one a quarter pint...
"I understand", says the bartender - and pours two pints.

MAT 132

8.7 Taylor and Maclaurin Series
8.8 Applications of Taylor Polynomials

Why we would ever want to express a function as a sum of **infinitely** many terms? Are we trying to make your life even more difficult ?

$f(x) = c_0 + c_1 x + c_2 x^2$ ... with convergence radius R>0.

- It is easy to integrate polynomials than certain functions. $\int e^{x^2} dx$
- It might help to find certain limits. $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$
- Often we do not have a formula for functions whose values we are trying to predict. In those cases, it is useful to know that we can approximate these functions by polynomials.

Suppose we have a function defined by a series

$f(x) = c_0 + c_1 x + c_2 x^2$... with convergence radius R>0.

- Find $f(0), f'(0), f''(0),.. ,f^{(n)}(0)$.
- Can you determine the values of the coefficients c_n in terms of the values of the function f?
- If $f(x)=e^x$, find an expression of f as a power series. What is the radius of convergence?

Suppose we have a function defined by a series

$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2$ ... with convergence radius R>0.

- Find $f(a), f'(a), f''(a),.. ,f^{(n)}(a)$.
- Can you determine the values of the coefficients c_n in terms of f?
- If $f(x)=\sin(x)$, can you find an expression of f as a power series centered at $\pi/2$?

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

The series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

is called the Taylor series of f at a.

When a=0, the series (below) is called the Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

In the case of the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that T_n is a polynomial n of degree n called the ***n*th degree Taylor polynomial** of *f* at *a*.

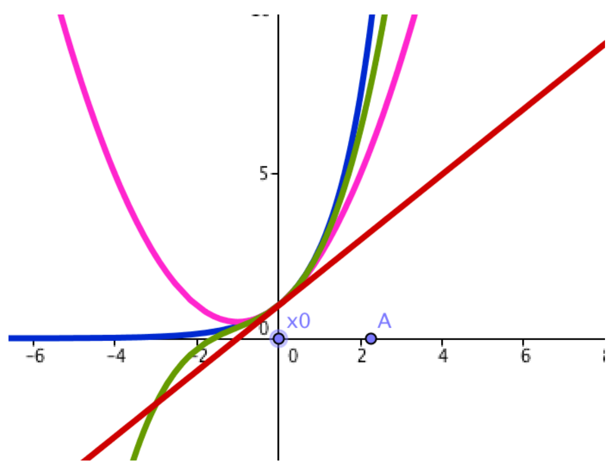
The remainder of order n of the Taylor series is

$$R_n(x) = f(x) - T_n(x)$$

Recall the Taylor series,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

In blue the graph of the function $f(x)=e^x$.
In brown, pink and green the Taylor polynomials $P_1(x)$, $P_2(x)$ and $P_3(x)$ respectively (centered at 0)



SUMMARY: Recall, given a function f, and a number a, **the Taylor polynomial of f of order n centered at a** and **remainder** are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$R_n(x) = f(x) - T_n(x)$$

- We want to approximate f(x) by $T_n(x)$ (for some n). Thus we need to know.
  - For a given n, how good is our approximation? Estimate how far are we from the actual value of the function.
  - If we want the "error" to be less than a given number, how large does n have to be?

The reminder ("error") is $R_n(x)=f(x)-T_n(x)$.

We discussed two methods for estimating $|R_n(x)|$ (each needs hypothesis)

- If the series is alternating, by the Alternating Series Estimation Theorem.
- If $|f^{(n)}(x)| \leq M$, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$

REVIEW The alternating series test

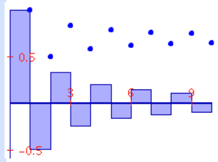
If we have a sequence { a_n },

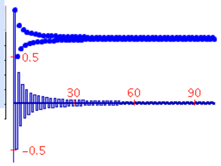
where $a_n > 0$, $a_n \geq a_{n+1}$, and $a_n \to 0$ when $n \to \infty$

then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges





Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i)}\ b_{n+1} \leq b_n \qquad \text{and} \qquad \text{(ii)}\ \lim_{n\to\infty} b_n = 0$$

then $$|R_n| = |s - s_n| \leq b_{n+1}$$

- Find the Taylor series for $f(x)=e^x$ at a=-3.
  - Use the series to approximate $e^{-3.1}$ correct to four decimal places.(you are given e^{-3})
- Find the Maclaurin series for f(x)=cos(x) and prove the series represents f(x) for all values of x.
- Find the Maclaurin series for f(x)=sin(x) and prove the series represents f(x) for all values of x.

b0=0.049
b1=0.0049
b2=0.00002

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R=1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R=\infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R=\infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R=\infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R=\infty$$

$$(1+x)^x = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} + \cdots \qquad R=\infty$$

- Find the Maclaurin series for $f(x)=(1+x)^k$, where k is any real number.

17 The Binomial Series If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

- Prove that the binomial series converges when |x|<1.
- Use the binomial series to expand $1/(1+x)^{0.5}$, as a power series. Use this to estimate 1/√1.1 correct to three decimal places.

b0=1
b1=0.-5
b2=0.00375
b3~0.00027

$$1 - \frac{1}{2}x + \frac{1\cdot 3}{2^2\cdot 2!}x^2 - \frac{1\cdot 3}{2^3\cdot 3!}x^3 + \cdots + (-1)^n \frac{1\cdot 3 \ldots (2n-1)}{2^n \cdot n!}x^n + \ldots$$

- 8.7.50 Use series to approximate the definite integral within the indicated accuracy correct to four decimal places. $\int_0^{0.5} x^2 e^{-x^2}\, dx$

- Use series to evaluate the limit $\lim_{x\to 0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5}$

- Evaluate the integral below, correct up to two decimal places. $\int_{\frac{1}{2}}^{1} \frac{\sin(x)}{x}\, dx$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

- The powers series representation of a function a given center “a” is unique.
- If f is a function and we know that f has a representation as a power series, then the Taylor series converges to f in the interval of convergence.
- If we are given a function f (we don’t know whether f has a representation as a power series,) the theorem below gives necessary conditions that imply that f has a representation as a power series:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$R_n(x) = f(x) - T_n(x)$$

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and
$$\lim_{n\to\infty} R_n(x) = 0$$
for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and
$$\lim_{n\to\infty} R_n(x) = 0$$
for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$R_n(x) = f(x) - T_n(x)$$

9 Taylor’s Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d$$

This implies that the Taylor series converges in this case. Why?