MAT 142
Problem Set #8
due in class on March 17, 2005

1. Apostol, section 10.9 # 1, 2, 5, 10
2. Apostol, section 10.10 # 3
3. Given a bounded sequence, a_n, define a new sequence, b_n, by

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

   (a) Prove that $\lim_{n\to\infty} b_n$ exists.
   This limit is called the *limit supremum* of a_n and is written as $\limsup_{n\to\infty} a_n$. Similarly, we can define the *limit infimum* as $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, where

$$c_n = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

   A similar proof also shows that the limit infimum always exists whenever a_n is bounded.
   (b) Use the squeeze principle to prove that if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_n = L$.
   (c) Compute the limit supremum and limit infimum of $a_n = (-1)^n$.
4. (a) Prove the following extension to the limit comparison test: If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
   (b) Give an example of two series where $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} b_n$ diverges.