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THE PERVERSE FILTRATION AND THE LEFSCHETZ HYPERPLANE THEOREM, II

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This paper is dedicated to the memory of Prakob Monkolchayut.

Abstract

The perverse filtration in cohomology and in cohomology with compact supports is interpreted, in terms of kernels of restriction maps to suitable subvarieties by using the Lefschetz hyperplane theorem and spectral objects. Various mixed-Hodge-theoretic consequences for intersection cohomology and for the decomposition theorem are derived.

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1. Introduction

Let Y be an affine variety of dimension n, let K be a bounded complex of sheaves of abelian groups on Y with constructible cohomology sheaves, and let $H^*(Y,K)$ be the (hyper)cohomology groups of Y with coefficients in K. For simplicity, in this introduction, we confine ourselves to affine varieties and to cohomology. In this paper, we prove analogous results for quasi-projective varieties, and for cohomology with compact supports. Fix an arbitrary embedding $Y \subseteq \mathbb{A}^N$ in affine space and let $Y_* = \{Y_{-n} \subseteq \ldots \subseteq Y_0 = Y\}$ be a general n-flag of linear sections of Y, i.e. $Y_{-i} = Y \cap \Lambda^{N-i}$, where $\{\Lambda^{N-n} \subseteq \ldots \subseteq \Lambda^N = \mathbb{A}^N\}$ is a general partial flag of linear affine subspaces of $\mathbb{A}^N$.

In [11], we showed that the (middle) perverse filtration on the cohomology groups $H^*(Y,K)$ (§3.1.5) can be described geometrically as follows: up to renumbering, *the perverse filtration coincides with the flag filtration*, i.e. the subspaces of the (middle) perverse filtration coincide with the kernels of the restriction maps $H^*(Y,K) \to H^*(Y_p, K_{|Y_p})$.

The purpose of this paper is twofold.

The former is to use Verdier's spectral objects to give, in §3.3, an alternative proof of the main result of [11], i.e. of the description of the perverse filtration in cohomology and in cohomology with compact supports using general flags. In order to do so, I introduce a new technique that, in the presence of suitable t-exactness, "realigns" a map of spectral sequences; see Lemma 3.2.3. I hope that this technique is of independent interest and will have further applications.

The latter is to use classical mixed Hodge theory and the description of the perverse filtration via flags to establish in a rather elementary fashion, in §4.3, a rather complete mixed-Hodge-theoretic package for the intersection cohomology groups of quasi-projective varieties and for the maps between them. For a survey on the decomposition theorem, see [13]. The results of this paper construct the mixed Hodge structures in a direct way, by using Deligne's theory, the decomposition theorem and various geometric constructions.

M. Saito proved these results in [21] by using mixed Hodge modules. While the mixed Hodge structures in §4.3 could be a priori different from the ones stemming from M. Saito's work, Theorem 4.3.5 asserts that the two structures coincide.

The mixed-Hodge theoretic results are stated in §4.3 and they are proved in §4.4 and §4.5. The strategy is to first prove the results in the case when the domain X of the map $f : X \to Y$ is nonsingular, and then to use the nonsingular case and resolution of singularities. This strategy is adapted from [8, 10], which deal with the case of projective varieties. In the projective case, the intersection cohomology groups coincide with the ones with compact supports, the Hodge structures are pure and key ingredients are the use of the intersection pairing on intersection cohomology and a *different* geometric description of the perverse filtration (see [8]), valid *only* in the projective case. In the present paper, we make a systematic use of the description of the perverse filtration via flags and, since we deal with noncompact varieties, the intersection pairing involves compact supports as well, and we must deal with intersection cohomology and with intersection cohomology with compact supports simultaneously. In the course of the proofs, care has to be taken to verify many mixed-Hodge-theoretic compatibilities, and I have included the details of the proof that I feel are not entirely a matter of routine.

Let me try to describe the meaning and the usefulness of the "realignment" Lemma 3.2.3 by discussing a simple situation; see also Example 3.2.1.

Let P be a perverse sheaf on $Y = \mathbb{A}^n$ and $i : Z \to Y$ be a general hyperplane. In general, while $P_{|Z}$ is not a perverse sheaf on Z, the shifted $P[-1]_{|Z}$ is. The natural adjunction map $P \to i_* i^* P$ yields the restriction map $r : H^*(Y, P) \to H^*(Z, P_{|Z})$. Since the cohomology perverse sheaves ${}^{\mathfrak{p}}\mathcal{H}^t(P) = 0$ for $t \neq 0$ and ${}^{\mathfrak{p}}\mathcal{H}^t(P_{|Z}) = 0$ for $t \neq -1$, the map of perverse spectral sequences $r_2^{st} : H^s(Y, {}^{\mathfrak{p}}\mathcal{H}^t(P)) \to H^s(Z, {}^{\mathfrak{p}}\mathcal{H}^t(P_{|Z}))$ that arises naturally by functoriality is the zero map. On the other hand, the Lefschetz hyperplane theorem for perverse sheaves implies that the restriction map r is injective in negative cohomological degrees.

The conclusion is that the map of spectral sequences that arises from restriction from Y to Z does not adequately reflect the geometry. In technical jargon, the issue is that i^* is not t-exact and the map of spectral sequences above has components r_2^{st} relating the "wrong" groups. This does not happen in the standard case, for i^* is always exact; see Remark 3.2.7.

The statement of the Lefschetz hyperplane theorem for perverse sheaves, coupled with the identities $P = {}^{\mathfrak{p}}\mathcal{H}^0(P)$ and $P_{|Z} = {}^{\mathfrak{p}}\mathcal{H}^{-1}(P_{|Z})[1]$, suggests that we should try and tilt the arrows r_2^{st} so that they start at a spot (s,t) but end at a spot $(s+1, t-1)$, thus being the arrows appearing in the Lefschetz hyperplane theorem. This tilting of the arrows can also be seen as a realignment of the pages of the target perverse spectral sequence (for Z) and we say that the target spectral sequence has been *translated* by one unit (see (3.9)).

The point is that this should be achieved in a coherent way, i.e. we are looking for a new map of spectral sequences, which has as source the original source, and as target the translate by one unit of the original target. This is precisely what Lemma 3.2.3 achieves in the more general context of spectral objects in t-categories. In the special case discussed above, the key point is that $i^*[-1]$ is suitably t-exact in view of the fact that the hyperplane H is general (see Remark 3.2.6), hence transversal to all the strata of a stratification for P. This explains why we need to choose the flag of linear sections Y_* to be general. The perverse spectral sequence for $H^*(Y, P)$ is now mapped nontrivially to the (-1)-translate of the perverse spectral sequence for $H^*(Z, P_{|Z})$.

The geometric description of the perverse filtration using flags (Theorem 3.3.1) is based on an iteration of this procedure for each element of the flag. By doing so, we obtain a compatible system of realigned maps of spectral sequences whose first pages have many zeroes (Artin vanishing theorem 2.0.1) and are connected by (mostly) monic arrows — epic if we are dealing with compact supports — (Lefschetz hyperplane theorem 2.0.3). The identification of the flag and perverse filtrations is then carried out by inspecting the pages of the realigned maps of spectral sequences (see §3.2.2 and the proof of Theorem 3.3.1).

2. Notation and background results

This paper is a sequel of [11] from which the notation is borrowed. However, this second part is independent of the first. Standard references for the language of derived categories and constructible sheaves are [5] and [19]; the reader may also consult [23].

A variety is a separated scheme of finite type over the field of complex numbers. In particular, we do not assume that varieties are irreducible. When dealing with intersection cohomology, it is convenient to assume irreducibility. However, it is not necessary to do so; see §4.6.

The term stratification refers to algebraic Whitney stratifications. Algebraic varieties and maps can be stratified.

Let Y be a variety. We denote by $\mathcal{D}_Y$ the bounded constructible derived category, i.e. the full subcategory of the derived category $D(Sh_Y)$ of the category of sheaves of abelian groups on Y consisting of bounded complexes K with constructible cohomology sheaves $\mathcal{H}^l(K)$ (though we do not pursue this, one can also get by with "weak constructibility" [19]). The objects are simply called constructible complexes.

The results of this paper that do not have to do with mixed Hodge theory, e.g. the ones in §3 concerning the geometric description of the perverse filtration, hold if we replace sheaves of abelian groups by sheaves of R-modules, where R is a Noetherian commutative ring with finite global dimension, e.g. $\mathbb{Z}$, a principal ideal domain, a field, etc.

The variants of these results for varieties over an arbitrary field of definition and for the various versions of étale cohomology also hold, with very similar proofs, and are left to the reader.

Given an algebraic map $f : X \to Y$, we have the usual derived functors f^*, $f_* := Rf_*$, $f_! := Rf_!$ and $f^!$ acting between $\mathcal{D}_X$ and $\mathcal{D}_Y$.

Given $K \in \mathcal{D}_Y$, we denote the (hyper)cohomology groups by $H^*(Y, K)$ and the (hyper)cohomology groups with compact supports by $H^*_c(Y, K)$.

A t-category (cf. [4, 19]) is a triangulated category $\mathcal{D}$ endowed with a t-structure. The truncation functors are denoted $\tau^a := \tau_{\geq a}$, $\tau_b := \tau_{\leq b}$, the cohomology functors $H:= H^0:=\tau^0 \circ \tau_0, H^l := H^0 \circ [l] = [l] \circ \tau^l \circ \tau_l = [l] \circ \tau_l \circ \tau^l$ have values in the abelian heart $\mathcal{C}$ of the t-category $\mathcal{D}$.

In this paper, we deal with the standard and with the middle-perversity t-structures on $\mathcal{D}_Y$. One word of caution: in the context of integer coefficients, Verdier Duality does not preserve middle perversity and there is no simple-minded exchange of cohomology with cohomology with compact supports. Because of this, at times we need to prove facts in cohomology and in cohomology with compact supports separately, although in our case the arguments run parallel, i.e. by inversion of the arrows. If one uses field coefficients, these repetitions can be avoided by invoking Verdier duality.

For the standard t-structure on $\mathcal{D}_Y$, we have the standard truncation functors, the cohomology functors are the usual cohomology sheaf functors $\mathcal{H}^*$ and the heart is equivalent to the category of constructible sheaves of $\mathbb{Z}$-modules on Y.

For the middle perversity t-structure, the truncation and cohomology functors are denoted by ${}^{\mathfrak{p}}\tau_{\geq l}$, ${}^{\mathfrak{p}}\tau_{\leq l}$, ${}^{\mathfrak{p}}\mathcal{H}^l$ and the heart is the abelian category $\mathcal{P}_Y \subseteq \mathcal{D}_Y$ of (middle) perverse sheaves of $\mathbb{Z}$-modules on Y.

An n-flag Y_* (see Example 3.1.5) on a variety Y is a sequence of closed subvarieties of Y:

$$\emptyset = Y_{-n-1} \subseteq Y_{-n} \subseteq Y_{-n+1} \subseteq \ldots \subseteq Y_{-1} \subseteq Y_0 = Y.$$

Typically, Y_{-i} will be the intersection of i hyperplane sections of a quasi-projective Y embedded in some projective space. For technical reasons, the embedding must be chosen to be affine (affine embeddings always exist). The "negative" indexing scheme for the elements of a flag serves the purposes of this paper.

All filtrations on abelian groups (and complexes) M etc., are decreasing, i.e. $F^iM \supseteq F^{i+1}M$ and finite, i.e. $F^{\ll 0}M = M$ and $F^{\gg 0}M = 0$. A filtration is said to be of type $[a, b]$, if $F^aM = M$ and $F^{b+1}M = 0$.

We shall consider the following filtration on cohomology as well as on cohomology with compact supports (see §3.1.5): the standard L_τ, the Leray L^f_τ, the perverse $L_{{}^{\mathfrak{p}}\tau}$ and the perverse Leray $L^f_{{}^{\mathfrak{p}}\tau}$ filtration. The indexing scheme differs slightly from the one in [11]. This is to serve the purposes of this paper, especially the proof of Propositions 3.2.10 and 3.2.11, where the indexing scheme employed in this paper places conveniently the spectral sequences in certain quadrants and facilitates the analysis.

Let $j : U \to Y \leftarrow Z : i$ be maps of varieties such that j is an open embedding and i is the complementary closed embedding. We have distinguished triangles and the long exact sequence of relative cohomology (in our situation we have that $j^* = j^!$ and $i_! = i_*$),

$$j_!j^!K \to K \to i_*i^*K \stackrel{[1]}{\to}, \tag{2.1}$$

$$\ldots \to H^*(Y, K_U) \to H^*(Y, K) \stackrel{r}{\to} H^*(Z, K_{|Z}) \to \ldots$$

$$i_!i^!K \to K \to j_*j^*K \stackrel{[1]}{\to}, \tag{2.2}$$

$$\ldots \to H^*_c(Z; i^!K) \stackrel{r'}{\to} H^*_c(Y, K) \to H^*_c(Y, Z, K) \to \ldots .$$

The complex $K_U = j_!j^!K$ on Y is not to be confused with $K_{|U}$ on U. Note also that $H^*(Y, K_U) = H^*(Y, Z, K)$.

The maps r and r' are called restriction and co-restriction maps.

In what follows, while the symbols are ambiguous, the formulæ are not. The slight abuse of notation is compensated by the simpler-looking formulæ. Given a Cartesian diagram of maps of varieties

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd} \tag{2.3}$$

there are the base change maps

$$g^*f_* \longrightarrow f_*g^*, \qquad g^!f_! \longleftarrow f_!g^!, \tag{2.4}$$

and the base change isomorphisms

$$g^*f_! \simeq f_!g^*, \qquad f_*g^! \simeq g^!f_*. \tag{2.5}$$

The base change maps (2.4) are isomorphisms if either one of the following conditions is met:

- f is proper; i.e. we have the base change theorem for the proper map f;
- f is locally topologically trivial over Y;
- g is smooth; i.e. we have the base change theorem for the smooth map g.

In fact, if f is proper, then $f_! = f_*$ and (2.5) implies (2.4); similarly, if g is smooth of relative dimension d, since then $g^! = g^*[2d]$ (cf. [19], Proposition 3.2.3). The remaining case follows easily from the Künneth formula.

The term "mixed Hodge (sub)structure" is abbreviated to MH(S)S.

The following is essentially due to M. Artin.

Theorem 2.0.1 (Cohomological dimension of affine varieties)**.** *Let Y be affine of dimension n, $P \in \mathcal{P}_Y$. Then*

$$H^r(Y,P) = 0, \quad \forall r \notin [-n,0], \qquad\qquad H^r_c(Y,P) = 0, \quad \forall r \notin [0,n].$$

Proof. See [19], Proposition 10.3.3 and Theorem 10.3.8. See also [23], corollaries 6.0.3 and 6.0.4. For the étale case, see [4], Théorème 4.1.1. □

Remark 2.0.2. In the context of this paper, a general hyperplane H is one chosen as follows. Pick any embedding of Y into projective space $\mathbb{P}$; take the closure $\overline{Y} \subseteq \mathbb{P}$; choose an algebraic Whitney stratification Σ of $\overline{Y}$ so that Y is a union of strata; choose, using the Bertini theorem, a hyperplane $\overline{H} \subseteq \mathbb{P}$ so that it meets transversally all the strata of Σ. Take $H := Y \cap \overline{H}$. For a discussion, see [12]; see also [11], §5.2.

Theorem 2.0.3 (Lefschetz hyperplane theorem)**.** *Let Y be quasi-projective of dimension n, $P \in \mathcal{P}_Y$. Let $H \subseteq Y$ be a general hyperplane section with respect to any embedding of Y into projective space and $J : (Y \setminus H) \to Y$ be the corresponding open immersion. We have:*

$$H^r(Y, J_!J^!P) = 0, \quad \forall r < 0, \qquad\qquad H^r_c(Y, J_*J^*P) = 0, \quad \forall r > 0.$$

Proof. This is due to several people: Goresky and MacPherson [17], Deligne (unpublished) and Beilinson, [3], Lemma 3.3. Deligne's and Beilinson's proofs are also valid in the étale case; see also [23], pp. 397–398. □

Remark 2.0.4. The Lefschetz hyperplane Theorem 2.0.3 implies that:

(1) the restriction map $H^r(Y,P) \to H^r(H,P_{|H})$ is an isomorphism for every $r \leq -2$ and it is injective for $r = -1$; similarly for $\mathcal{F}$, and
(2) the co-restriction map $H^r_c(H, i^!P) \to H^r_c(Y,P)$ is an isomorphism for $r \geq 2$ and surjective for $r = 1$.

To my knowledge, there is no analogue of (2). for constructible sheaves $\mathcal{F}$. Note also that since H is general, $i^*P[-1] = i^!P[1]$ (cf. [8], Lemma 3.5.4(b), or [23], p. 321) is perverse on H.

In [12], section 3.3.2, I incorrectly stated that the Lefschetz hyperplane theorem requires to first choose an affine embedding of Y into projective space.

Remark 2.0.5. For completeness, let us mention that Theorems 2.0.1 and 2.0.3 admit well-known versions for constructible sheaves; see [12], Appendix and also [23].

The following is due to J.P. Jouanolou.

Proposition 2.0.6. *Let Y be a quasi-projective variety. There is a natural number d and a Zariski locally trivial $\mathbb{A}^d$-fibration $\pi : \mathcal{Y} \to Y$ with affine transition functions and affine total space $\mathcal{Y}$.*

Proof. See [18]; see also [12]. □

Remark 2.0.7. There is no canonical choice for the fibration. One can arrange for $d = \dim Y$ but, in general, not less, e.g. $Y = \mathbb{P}^n$.

3. The perverse filtrations via spectral objects

The goal of this section is to prove the results in §3.3. This is done using the preparatory results on spectral objects and spectral sequences in §3.2. Spectral objects are discussed in §3.1.

3.1. Spectral objects and spectral sequences.

3.1.1. Spectral objects in a triangulated category. Spectral objects have been introduced by Verdier. A good reference is [16].

Let $\mathcal{D}$ be a triangulated category. A *spectral object* in $\mathcal{D}$ is the data:

(1) a family of objects X_{pq} of $\mathcal{D}$ indexed by pairs $p \leq q \in \mathbb{Z}$,

(2) for $p' \leq p$, $q' \leq q$, a morphism $X_{pq} \to X_{p'q'}$,

(3) for $p \leq q \leq r$, a morphism $\partial : X_{pq} \to X_{qr}[1]$ called *coboundary*

subject to the requirement that:

(a) the morphisms (2) define a *contravariant* functor from the category of ordered pairs (p,q), with $p \leq q$, to $\mathcal{D}$,

(b) for $p \leq q \leq r$, $p' \leq q' \leq r'$, and $p' \leq p$, $q' \leq q$, $r' \leq r$, the diagram

$$\begin{tikzcd} X_{pq} \arrow[r, "\partial"] \arrow[d] & X_{qr}[1] \arrow[d] \\ X_{p'q'} \arrow[r, "\partial"] & X_{q'r'}[1] \end{tikzcd}$$

of morphisms from (2) and (3) is commutative,

(c) for $p \leq q \leq r$ the triangle

$$X_{qr} \longrightarrow X_{pr} \longrightarrow X_{pq} \xrightarrow{\partial} X_{qr}[1]$$

is distinguished.

There is an obvious notion of morphism of spectral objects and spectral objects in $\mathcal{D}$ form a category.

The axioms imply $X_{pp} = 0$ so that, on the (p,q)-plane, the display occurs on and above the line $q = p + 1$.

We work exclusively with *bounded* spectral objects, i.e. objects for which $X_{p,p+1} = 0$ for $|p| \gg 0$. If $X_{p,p+1} = 0$ for $p \neq [a,b]$, then we say the spectral object has *amplitude in* $[a,b]$. In this case: (i) all X_{pq} with $p < a$ and $q > b$ are isomorphic to one another via the maps (2) and we denote them by $X_{-\infty,\infty}$, (ii) all the $X_{p,q}$, with p fixed and $q > b$, are isomorphic via the maps (2) and we denote them by $X_{p,\infty}$.

Up to applications of the translation functor $C \mapsto C[1]$, one may choose to consider only the case of amplitude in $[0, b-a]$, or $[a-b, 0]$. In the former case we have that $X_{0,q} \simeq X_{-1,q} \simeq X_{-2,q} \ldots$, for every q and the essential part of the display is a triangle in the first quadrant.

Giving a spectral object with amplitude in $[0,1]$ is the same as giving a distinguished triangle. Amplitude in $[0,2]$ corresponds to an octahedron diagram (as in the octahedron axiom for triangulated categories), etc. A spectral object is therefore a suitably compatible system of triangles, octahedrons, etc.

Example 3.1.1 (Filtered complexes)**.** Let $\mathcal{A}$ be an abelian category and (K,F) be a filtered complex with finite filtration. By setting $X_{pq} := F^pK/F^qK$, one gets a bounded spectral object in the derived category $D(\mathcal{A})$. In other words, an object of the finite filtered derived category $DF(\mathcal{A})$ yields a spectral object in $D(\mathcal{A})$.

Example 3.1.2 (Sequence of maps)**.** Let $\mathcal{A}$ be an abelian category with enough injectives and

$$\ldots \longrightarrow K_{i+1} \longrightarrow K_i \longrightarrow K_{i-1} \longrightarrow \ldots$$

be a sequence of morphisms in $D^+(\mathcal{A})$, with $K_i = 0$, for $i \gg 0$ and $K_{i+1} \simeq K_i$, for $i \ll 0$. There exists a filtered complex, with finite filtration, (K,F) in $D^+(\mathcal{A})$, such that the given sequence is isomorphic to the sequence of sub-complexes F^iK (see [4], p. 77). In the cases we use in this paper, this correspondence is canonical, i.e. defined up to unique isomorphism in the filtered derived category. In view of Example 3.1.1, we shall speak of the spectral object in $D^+(\mathcal{A})$ associated with a sequence of morphisms as above.

Example 3.1.3 (Truncation)**.** Let $\mathcal{D}$ be a t-category with truncation functors τ_i, τ^i and cohomology functors $H^i = [i] \circ \tau^i \circ \tau_i$. An object X of $\mathcal{D}$ yields a spectral object in $\mathcal{D}$ by setting:

$$X_{pq} := \tau^{-q+1} \tau_{-p} X.$$

This correspondence is functorial. If X is bounded, then $X_{p,\infty} = \tau_{-p} X$. This choice of indexing leads to a decreasing filtration in cohomology (cf. §3.1.3, (3.4)). One has

$$X_{p,p+1} = \tau^{-p} \tau_{-p} X = H^{-p}(X)[p].$$

Cohomological amplitude $[0,n]$ here means that $H^l(X) = 0$ for $l \notin [-n, 0]$.

Remark 3.1.4. The category $\mathcal{D}_Y$ of bounded constructible complexes on a variety Y admits the standard t-structure, i.e. the one associated with the natural truncation functors. It also admits the distinct t-structures associated with distinct perversities. Each t-structure yields spectral objects as in Example 3.1.3 which are isomorphic to the spectral objects we obtain by virtue of Example 3.1.2 applied to the sequence of truncation maps associated with the t-structure: $\ldots \tau_{-i-1} X \longrightarrow \tau_{-i} X \longrightarrow \ldots$

Example 3.1.5 (Sequence of closed subvarieties)**.** An *n-flag* Y_* on the variety Y is defined to be a sequence

$$\emptyset = Y_{-n-1} \subseteq Y_{-n} \subseteq Y_{-n+1} \subseteq \ldots \subseteq Y_{-1} \subseteq Y_0 = Y$$

of closed subvarieties of Y. The indexing scheme is chosen as to serve the needs of this paper and obtain filtrations F_{Y_*} of type $[-n,0]$ in cohomology and of type $[0,n]$ in cohomology with compact supports. Let

$$j_p : Y \setminus Y_p \longrightarrow Y, \qquad i_p : Y_p \longrightarrow Y, \qquad k_p : Y_p \setminus Y_{p-1} \longrightarrow Y$$

be the corresponding embeddings.

Let $K \in \mathcal{D}_Y$. Note that $j_{p!} j_p^* K = K_{Y - Y_{p-1}}$ and that $i_{p*} i_p^! K = R\Gamma_{Y_p} K$.

Without loss of generality, we assume that K is injective so that $R\Gamma_{Y_p} K = \Gamma_{Y_p} K$; this fact allows us, in particular, to form the quotients in (3.2).

We have the two spectral objects K_{pq} and $K^!_{pq}$ associated with the two filtered complexes

$$(3.1) \qquad (K, F_{Y_*}), \qquad F^p_{Y_*} K := K_{Y - Y_{p-1}}, \qquad K_{pq} := F^p_{Y_*} K / F^q_{Y_*} K,$$

$$(3.2) \qquad (K, G_{Y_*}), \qquad G^p_{Y_*} K := R\Gamma_{Y_{-p}} K, \qquad K^!_{pq} := G^p_{Y_*} K / G^q_{Y_*} K.$$

The spectral object K_{pq} has amplitude in the interval $[-n, 0]$ and $K^!_{pq}$ in $[0,n]$ and the resulting filtrations have type $[-n,0]$ and $[0,n]$, respectively.

3.1.2. Spectral objects in an abelian category with translation functor. Let $\mathcal{B}$ be an abelian category and $\mathcal{A}$ be the associated category of graded objects. The abelian category $\mathcal{A}$ is naturally endowed with the translation functor, denoted [1], which is an autoequivalence.

A spectral object in $\mathcal{A}$ is defined analogously to one in $\mathcal{D}$ with the following adapted axioms:
($1'$) the objects X_{pq} are in fact collections $\{X^n_{pq}\}_{n\in\mathbb{Z}}$ in $\mathcal{B}$;
($3'$) $\partial : X^n_{pq} \to X^{n+1}_{qr}$;
(c') there is a long exact sequence

$$\ldots \longrightarrow X^n_{pr} \longrightarrow X^n_{pq} \xrightarrow{\partial} X^{n+1}_{qr} \longrightarrow X^{n+1}_{pr} \longrightarrow \ldots.$$

Let $\mathcal{D}$ be a triangulated category, $\mathcal{B}$ be an abelian category and $T : \mathcal{D} \to \mathcal{B}$ be a cohomological functor. Denote by $T^\bullet : \mathcal{D} \to \mathcal{A}$ the induced functor: $T^n(X) := T(X[n])$.

In either of the categories $\mathcal{D}$ and $\mathcal{A}$, there is the obvious notion of morphism of spectral objects $X_{pq} \to X'_{pq}$. Spectral objects in $\mathcal{D}$ and in $\mathcal{A}$ form categories and $T^\bullet$ is a functor transforming spectral objects in $\mathcal{D}$ into spectral objects in $\mathcal{A}$: if X is a spectral object in $\mathcal{D}$, then it is immediate to verify that we obtain a spectral object $T(X)$ in $\mathcal{A}$ by setting

$$T(X)^n_{pq} := T^n(X_{pq}).$$

3.1.3. Spectral objects and spectral sequences. Let $T : \mathcal{D} \to \mathcal{B}$ and $T^\bullet : \mathcal{D} \to \mathcal{A}$ be as above. Then T also induces a functor into the category of spectral sequences. If X is a bounded spectral object in $\mathcal{D}$, then there is the object $X_{-\infty,\infty}$, canonically isomorphic to X, and the spectral sequence $E_1(T(X))$:

$$E_1^{pq}(T(X)) := T^{p+q}(X_{p,p+1}) \Longrightarrow T^{p+q}(X) := T^{p+q}(X_{-\infty,\infty}). \tag{3.3}$$

The associated decreasing and finite filtration is
(3.4)

$$F^pT^u(X) = \mathrm{Im}\,\{T^u(X_{p,\infty})\} \subseteq T^u(X), \qquad F^pT^{p+q}/F^{p+1}T^{p+q} = E^{p,q}_\infty.$$

3.1.4. Spectral sequences: re-numeration, the L filtration, translation. Let $E_1^{pq} \Longrightarrow T^{p+q}$ be a spectral sequence as in §3.1.3 with abutment the filtration F.

In view of Propositions 3.2.10 and 3.2.11, we introduce a different indexing scheme. The *re-numeration* $\mathcal{E}_{r+1}$ of E_r, is the *same* spectral sequence, with a different indexing scheme:

$$\mathcal{E}^{st}_{r+1} := E_r^{-t,s+2t}, \quad r \geq 1, \qquad \mathcal{E}^{st}_2 \Longrightarrow T^{s+t}. \tag{3.5}$$

One still has the filtration F, but also has the L-*filtration* (which we adopt in this paper)

$$F^pT^u =: L^{p+u}T^u, \qquad L^sT^{s+t}/L^{s+1}T^{s+t} = \mathcal{E}^{st}_\infty. \tag{3.6}$$

The following relations are easily verified:

$$F^p(T^u(X[d])) = F^{p-d}(T^{u+d}(X)), \qquad L^p(T^u(X[d])) = L^p(T^{u+d}(X)). \tag{3.7}$$

Example 3.1.6. If one re-numbers the usual Grothendieck spectral sequence for cohomology $E_1^{pq} = H^{p+q}(Y, \mathcal{H}^{-p}(K)[p]) \Longrightarrow H^{p+q}(Y,K)$, which has

$$F^pH^j(Y,K) = \operatorname{Im}\{H^*(Y, \tau_{\leq -p}K) \longrightarrow H^*(Y,K)\},$$

then one obtains $\mathcal{E}_2^{st} = H^s(Y, \mathcal{H}^t(K)) \Longrightarrow H^{p+q}(Y,K)$ with decreasing L-filtration

$$L^sH^j(Y,K) = \operatorname{Im}\{H^j(Y, \tau_{\leq -s+j}(K)\}.$$

Note that the re-numbered Grothendieck spectral sequence is in the quadrants I and IV.

The l-*translate* $G(l)$ of a filtration G is defined by setting

$$G(l)^s := G^{l+s}. \tag{3.8}$$

The l-*translate* $E_r(l)$ of a spectral sequence E_r with abutment a filtration F is defined as follows:

$$E_r(l)^{pq} := E_r^{p+l,q-l} \Longrightarrow T^{p+q}. \tag{3.9}$$

The associated filtrations satisfy,

$$F(E_1(l)) = F(E_1)(l). \tag{3.10}$$

This formula also holds for the associated L filtration and one has $F(l)^pT^u = L(l)^{p+u}T^u$.

3.1.5. The spectral sequence associated with a t-structure: standard and perverse spectral sequences and filtrations. Let $\mathcal{D}$ be a t-category, let $T : \mathcal{D} \to \mathcal{B}$ be a cohomological functor and let $T^\bullet : \mathcal{D} \to \mathcal{A}$ be the associated graded functor. Given $X \in \mathcal{D}$, denote also by X the associated spectral object via Example 3.1.3. Note that $T(X)$ now has two different meanings: $T(X) \in \mathcal{B}$ and $T(X)$ the spectral object in $\mathcal{A}$. I hope this does not generate confusion.

Recall the formula $X_{p,p+1} = H^{-p}(X)[p]$. Assume that $H^l(X) = 0$, for $|l| \gg 0$, i.e. that the spectral object X is bounded. There is the spectral sequence $E_1(T(X))$ as in §3.1.3:

$$E_1^{pq} := T^{p+q}(H^{-p}(X)[p]) = T^{2p+q}(H^{-p}(X)) \Longrightarrow T^{p+q}(X).$$

If we re-number as in (3.5), then we have $\mathcal{E}_2(T(X))$:

$$\mathcal{E}_2^{s,t} T^s(H^t(X)) \Longrightarrow T^{s+t}(X).$$

For the associated filtrations, we have $F^p T^u(X) = \operatorname{Im}\{T^u(\tau_{-p}X) \to T^u(X)\} \subseteq T^u(X)$, and we have the equalities (3.6).

In this paper, we work with the standard and with the middle perversity t-structures on $\mathcal{D}_Y$, denoted by τ and ${}^{\mathfrak{p}}\tau$ respectively, and with the cohomology and compactly supported cohomology functors, i.e. with the special cases when $T = H^0(Y, -)$, or $T = H^0_c(Y, -)$.

Let Y be a variety and $K \in \mathcal{D}_Y$. We have the following spectral sequences with associated filtrations:

$$\mathcal{E}_2^{st}(K, \tau) := H^s(Y, \mathcal{H}^t(K)) \Longrightarrow (H^{s+t}(Y, K), L_\tau), \tag{3.11}$$

$$\mathcal{E}_2^{st}(K, {}^{\mathfrak{p}}\tau) := H^s(Y, {}^{\mathfrak{p}}\mathcal{H}^t(K)) \Longrightarrow (H^{s+t}(Y, K), L_{{}^{\mathfrak{p}}\tau}). \tag{3.12}$$

We have analogous sequences ${}_c\mathcal{E}$ for compactly supported cohomology $H^*_c(Y, K)$.

The spectral sequence (3.11) is called the *standard* (or Grothendieck) *spectral sequence* and the associated filtration of type L (versus type F) is called the *standard filtration* and is denoted L_τ. The spectral sequence (3.12) is called the *perverse spectral sequence* and the associated filtration of type L is called the *perverse filtration* and is denoted by $L_{{}^{\mathfrak{p}}\tau}$.

Let $f : X \to Y$ be a map of varieties and let $C \in \mathcal{D}_X$. We have the Leray and the perverse Leray spectral sequences with associated Leray and perverse-Leray filtrations:

(3.13)

$$\mathcal{E}_2^{st}(f_*C, \tau) \Longrightarrow (H^{s+t}(X, C), L^f_\tau), \qquad {}_c\mathcal{E}_2^{st}(f_!C, \tau) \Longrightarrow (H_c^{s+t}(X, C), L^f_\tau);$$

(3.14)

$$\mathcal{E}_2^{st}(f_*C, {}^{\mathfrak{p}}\tau) \Longrightarrow (H^{s+t}(X, C), L^f_{{}^{\mathfrak{p}}\tau}), \qquad {}_c\mathcal{E}_2^{st}(f_!C, {}^{\mathfrak{p}}\tau) \Longrightarrow (H_c^{s+t}(X, C), L^f_{{}^{\mathfrak{p}}\tau}).$$

Remark 3.1.7. The standard spectral sequences are in the quadrants I and IV. Let Y be affine of dimension n. Theorem 2.0.1 implies that the standard filtration is of type $[0, n]$ in cohomology and of type $[n, 2n]$ in cohomology with compact supports and the corresponding standard spectral sequences $\mathcal{E}_r^{s,t}$, ${}_c\mathcal{E}_r^{s,t}$ live in the columns $s \in [0, n]$ and $s \in [n, 2n]$, respectively. The perverse filtration has types $[-n, 0]$ and $[0, n]$, respectively, and the perverse spectral sequences are in the quadrants II and III (cohomology), with columns $s \in [-n, 0]$, and I and IV (cohomology with compact supports), with columns $s \in [0, n]$. If Y is a variety of dimension n covered by $1 + t$ affine open sets, then the four types for the filtrations above are $[0, n + t]$, $[n - t, 2n]$, $[-n, t]$

and $[-t, n]$, respectively and the spectral sequences live in the corresponding columns.

3.1.6. The spectral sequence associated with a flag of closed subspaces. Let Y be a variety, let $K \in \mathcal{D}_Y$ and let $Y_* = \{\emptyset \subseteq Y_{-n} \subseteq \ldots \subseteq Y_0 = Y\}$ be an n-flag as in Example 3.1.5. There are the two spectral objects $K_{p,q}$ and $K^!_{pq}$ in $\mathcal{D}_Y$. Their amplitudes are in $[-n, 0]$ and in $[0, n]$, respectively. If we apply the cohomology functors $H^*(Y, -)$ to the first one and $H^*_c(Y, -)$ to the second one, then we get two spectral objects whose associated spectral sequences are the classical spectral sequences for the filtration of Y into closed subspaces:

- the spectral sequence for cohomology and the associated filtration are $E_1(K, Y_*)$:

$$E_1^{p,q} = H^{p+q}(Y, K_{Y_p - Y_{p-1}}) \Longrightarrow H^{p+q}(Y, K), \tag{3.15}$$

$$F^p_{Y_*} H(Y, K) = \ker\{H^*(Y, K) \longrightarrow H^*(Y_{p-1}, K_{p-1})\}; \tag{3.16}$$

- the spectral sequence and filtration for cohomology with compact supports are ${}_cE_1(K, Y_*)$:

$${}_cE_1^{p,q} = H_c^{p+q}(Y, R\Gamma_{Y_{-p} - Y_{-p-1}} K) \Longrightarrow H_c^{p+q}(Y, K), \tag{3.17}$$

$$G^p_{Y_*} H^*_c(Y, K) = \mathrm{Im}\,\{H^*_c(Y_{-p}, i^!_{-p} K) \longrightarrow H^*_c(Y, K)\}. \tag{3.18}$$

Remark 3.1.8. Note that $F^{-n}_{Y_*} H^*(Y, K) = H^*(Y, K)$ and $F^1 H^*(Y, K) = 0$, i.e. the filtration F_{Y_*} in cohomology has type $[-n, 0]$. Analogously, G_{Y_*} has type $[0, n]$.

3.2. Preparatory results.

3.2.1. Translating spectral objects. A map $X \to Y$ in a t-category $\mathcal{D}$ yields a map of spectral objects $X_{pq} \to Y_{pq}$. By applying a cohomological functor $T : \mathcal{D} \to \mathcal{B}$ and by recalling the re-numeration (3.5), we get a morphism of spectral sequences

$$\mathcal{E}_r^{st}(X) \longrightarrow \mathcal{E}_r^{st}(Y), \quad r \geq 2.$$

Example 3.2.1. Let $i : pt \to \mathbb{A}^1$. We have the adjunction morphism

$$\mathbb{Q}_{\mathbb{A}^1}[1] \longrightarrow i_* i^* \mathbb{Q}_{\mathbb{A}^1}[1] = \mathbb{Q}_{pt}[1].$$

We consider the standard and the middle-perversity t-structures. The map of spectral sequences associated with the standard t-structure is an isomorphism of spectral sequences and it induces an isomorphism on the filtered abutments. On the other hand, middle-perversity yields the zero map of spectral sequences. Moreover, we have

$$F^0 H^{-1}(\mathbb{A}^1, \mathbb{Q}_{\mathbb{A}^1}[1]) = \mathbb{Q}, \quad F^1 H^{-1}(\mathbb{A}^1, \mathbb{Q}_{\mathbb{A}^1}[1]) = 0, \quad Gr^0_F = \mathbb{Q}, \quad Gr^1_F = 0,$$
$$F^1 H^{-1}(pt, \mathbb{Q}_{pt}[1]) = \mathbb{Q}, \quad F^2 H^{-1}(pt, \mathbb{Q}_{\mathbb{A}^1}[1]) = \mathbb{Q}, \quad Gr^0_F = 0 \quad Gr^1_F = \mathbb{Q}.$$

Similarly, for the L filtrations for which we have, respectively: $Gr_L^{-1} = \mathbb{Q}$, $Gr_L^0 = 0$ and $Gr_L^0 = \mathbb{Q}$, $Gr_L^1 = 0$. In particular, a mere re-numbering of the spectral sequences will not yield an isomorphism. The reason for this different behavior is explained by Lemma 3.2.3: the functor i^* is t-exact for the standard t-structure, but it is not for the middle-perversity t-structure.

In the example above one could re-index the filtration on $H^{-1}(pt, \mathbb{Q}_{pt}[1])$ so that, for example, $L^{-1}H^{-1}(\mathbb{Q}_{\mathbb{A}^1}[1]) \simeq L^{-1}H^{-1}(\mathbb{Q}_{pt}[1])$. One can do so, but may also care to define a morphism of spectral sequences compatible with this objective. In this case, this is possible, due to the presence of suitable t-exactness for $i^*[-1]$ with respect to $\mathbb{Q}_{\mathbb{A}^1}[1]$; see Remark 3.2.6.

The necessity of considering a translated filtration on, say, the second spectral sequence while at the same time retaining a map of spectral sequences, has lead us to the notion of translated spectral objects. We need to have a meaningful map of spectral sequences after this translation has taken place. Lemma 3.2.3 is a criterion for reaching this objective.

Recall the notions (3.8) of translated filtration, (3.9) of translated spectral sequence and the effect (3.10) of translation on the abutted F and L filtrations.

Definition 3.2.2. Let $X = \{X_{pq}\}$ be a spectral object in $\mathcal{D}$ (let $X = \{X^n_{pq}\}$ be a spectral object in $\mathcal{A}$, respectively). Let $l \in \mathbb{Z}$ and define the *translation of* X *by* l to be the spectral object $X(l)$:

$$X(l)_{pq} := X_{p+l,q+l}, \qquad (X(l)^n_{pq} := X^n_{p+l,q+l}, \quad \text{respectively}).$$

In the presence of a cohomological functor $T : \mathcal{D} \to \mathcal{B}$, a spectral object X in $\mathcal{D}$ gives the spectral object $T(X)$ in $\mathcal{A}$ and we have $(T(X)(l))^n_{pq} = T^n(X_{p+l,q+l})$.

We have the following simple equalities

$$E_1(T(X(l))) = E_1(T(X))(l) \Longrightarrow (T(X), L(l)). \tag{3.19}$$

Let $d \in \mathbb{Z}$ and $v : \mathcal{D} \longrightarrow \mathcal{D}'$ be a functor of t-categories, i.e. v is additive, it commutes with translations and it sends distinguished triangles into distinguished triangles. We say that $v \circ [d]$ is t-exact if

$$v(\tau^a(X)) = \tau^{a+d}(v(X)), \qquad v(\tau_a(X)) = \tau_{a+d}(v(X)) \tag{3.20}$$

or, equivalently, if:

$$[v \circ [d], \tau^a] = [v \circ [d], \tau_a] = [v \circ [d], H^a] = 0.$$

Lemma 3.2.3. *Let*

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "u^\bullet"] \arrow[dr, "T"'] & & \mathcal{D}' \arrow[dl, "T'"] \\ & \mathcal{B} & \end{tikzcd}$$

be a commutative diagram of functors, with $\mathcal{D}$ and $\mathcal{D}'$ t-categories and $\mathcal{B}$ abelian such that:

(1) *$u^\bullet[d]$ is t-exact, for a fixed $d \in \mathbb{Z}$,*
(2) *T and T' are cohomological,*
(3) *there exists $u_\bullet : \mathcal{D}' \to \mathcal{D}$ such that $(u^\bullet, u_\bullet)$ is an adjoint pair and*
(4) *$T \circ u_\bullet = T'$; in particular, $Tu_\bullet u^\bullet = T'u^\bullet$.*

Let $X \in \mathcal{D}$ and $X_{pq} = \tau^{-q+1}\tau_{-p}X$ be the associated spectral object as in Example 3.1.3. *Then there is a morphism of spectral objects in $\mathcal{A}$:*

$$T(X) \longrightarrow [T'(u^\bullet X)](-d)$$

yielding a morphism of spectral sequences

$$E_1(T(X)) \longrightarrow E_1(T'(u^\bullet X)(-d))$$

with resulting filtered map

$$(T^{p+q}(X), L) \longrightarrow ({T'}^{p+q}(u^\bullet X), L(-d)).$$

Analogously, if $(u_\bullet, u^\bullet)$ is an adjoint pair, then the statement with arrows reversed holds true.

Proof. We prove the first statement. The proof of the second one is identical.

By adjointness, there is the morphism of functors $a : Id \to u_\bullet u^\bullet$ inducing a map

$$T^n(\tau^{-q+1}\tau_{-p}X) \longrightarrow T^n(u_\bullet u^\bullet \tau^{-q+1}\tau_{-p}X) \stackrel{(4)}{=} {T'}^n(u^\bullet \tau^{-q+1}\tau_{-p}X).$$

On the other hand, by t-exactness (3.20) and by the definition of translation 3.2.2, we get the natural identifications

$${T'}^n(u^\bullet \tau^{-q+1}\tau_{-p}X) = {T'}^n(\tau^{-q+d+1}\tau_{-p+d}u^\bullet X) = [T'(u^\bullet X)(-d)]^n_{pq}.$$

We thus get a natural map induced by the adjunction map $a : X \to u_\bullet u^\bullet X$:

$$T(X)^n_{pq} = T^n(\tau^{-q+1}\tau_{-p}X) \longrightarrow {T'}^n(\tau^{-q+d+1}\tau_{-p+d}u^\bullet X) = T'(u^\bullet X)(-d)_{pq}.$$

Since a is a morphism of functors, checking that the map above induces a morphism of spectral objects in $\mathcal{A}$ reduces to a formal verification. □

Remark 3.2.4. Example 3.2.1 shows that the map of spectral sequences arising via functoriality from the adjunction map $X \to u_\bullet u^\bullet X$ is *not* the one of Lemma 3.2.3 which needs the extra input of t-exactness. Note also that the realigned map of spectral sequences $E_1 \to E_1(-d)$ does not correspond to a realigned map of spectral objects $X \to u_\bullet u^\bullet X[-d]$.

Remark 3.2.5. The map of spectral sequences of Lemma 3.2.3 is, a bit more explicitly:

$$T^{p+q}(H^{-p}(X)[p]) \longrightarrow {T'}^{p+q}(H^{-p+d}(u^\bullet X)[p-d]).$$

Remark 3.2.6. The functor $i^*[-1]$ of Example 3.2.1 is not t-exact for the middle perversity t-structures since, for example, $i^*\mathbb{Q}_{pt}[-1]$ is not perverse. However, it is t-exact when applied to complexes, such as $\mathbb{Q}_{\mathbb{A}^1}[1]$ which are constructible with respect to a stratification for which the inclusion $i : pt \to \mathbb{A}^1$ is a normally nonsingular inclusion of codimension one. Such complexes form a t-subcategory and $i^*[-1]$ is t-exact when acting on such complexes. It follows that one can apply Lemma 3.2.3 with $u = i^*$, $d = -1$ and get the desired isomorphism of spectral sequences.

Remark 3.2.7. Lemma 3.2.3 implies the usual functoriality of the Leray spectral sequence with respect to square commutative diagrams (2.3). One only needs to observe that g^* is exact in the usual sense, i.e. t-exact for the standard t-structure. In fact, we have the adjunction map $f_*C \to g_*g^*f_*C$ and the natural map $g^*f_*C \to f_*g^*C$. By applying the lemma to the first and functoriality to the second, we obtain $H^s(Y, R^tf_*C) \to H^s(Y', \mathcal{H}^t(g^*f_*C)) \to H^s(Y', R^sf_*(g^*C))$.

3.2.2. Inductive behavior of spectral sequences. In this section we prove Proposition 3.2.10 and its analogue Proposition 3.2.11. These propositions are used in the proof of Theorems 3.3.1 and 3.3.5, i.e. the proof of the geometric description of the perverse and perverse Leray filtrations in the quasi-projective case.

The following lemma is spelled-out for the reader's convenience.

Lemma 3.2.8. *Let*

$$\begin{tikzcd} E' \arrow[r, "d'_E"] \arrow[d, "\phi'"] & E \arrow[r, "d''_E"] \arrow[d, "\phi"] & E'' \arrow[d, "\phi''"] \\ F' \arrow[r, "d'_F"] & F \arrow[r, "d''_F"] & F'' \end{tikzcd}$$

be a commutative diagram in an Abelian category such that $d^2 = 0$, ϕ' *epic,* ϕ *monic (*ϕ'' *monic,* ϕ *epic, respectively). Then the induced map on* d-*cohomology*

$$H_E \stackrel{H^*(\phi)}{\longrightarrow} H_F$$

is monic (epic, respectively).

Proof. Since ϕ is monic, the induced map on kernels is monic and since ϕ' is epic, the d'-images map isomorphically onto each other. This proves the first statement. As to the second one, since ϕ is epic and ϕ'' is monic, the induced map on kernels is epic and we are done. $\square$

The following lemma is the key step in the proof of the propositions that follow. The first statement is used to study cohomology and the second one to study cohomology with compact supports. We state it in the way we shall use it, i.e. using the $(\mathcal{E}, L)$-notation of (3.5) and (3.6), i.e. $L^s H^{s+t}/L^{s+1}H^{s+t} = \mathcal{E}_\infty^{s,t}$.

Lemma 3.2.9. (1) *Let $\phi : \mathcal{E} \to \mathcal{F}$ be a morphism of bounded* II/III-*quadrants spectral sequences with abutments $H_{\mathcal{E}}$ and $H_{\mathcal{F}}$. Assume that $\phi_2^{pq} : \mathcal{E}_2^{pq} \to \mathcal{F}_2^{pq}$ is an isomorphism for every $p \leq -2$, injective for $p = -1$ and zero for $p = 0$. Then ϕ_∞^{pq} is injective for every $p \leq -1$ and*

$$\operatorname{Ker}\{H^*(\phi) : H_{\mathcal{E}} \to H_{\mathcal{E}}\} = L^0 H_{\mathcal{E}}.$$

(2) *Let $\phi : \mathcal{E} \to \mathcal{F}$ be a morphism of bounded* I/IV-*quadrants spectral sequences. Assume that $\phi_2^{pq} : \mathcal{E}_2^{pq} \to \mathcal{F}_2^{pq}$ is an isomorphism for every $p > 1$, surjective for $p = 1$ and zero for $p = 0$. Then ϕ_∞^{pq} is surjective for every $p \geq 1$ and*

$$L^1 H_F = \operatorname{Im}\{H^*(\phi) : H_{\mathcal{E}} \to H_{\mathcal{F}}\}.$$

Proof. In either case, $\phi_r^{0,q} = 0$, for every q, r.

Let us prove (1).

Claim. Let $r \geq 2$ be fixed. Then

$$\mathcal{E}_r^{p,\bullet} \longrightarrow \mathcal{F}_r^{p,\bullet}$$

is an isomorphism $\forall p \leq -r$ and is monic for $\forall p \leq -1$.

The proof is by induction on r. If $r = 2$, then the CLAIM is true by hypothesis.

Assume we have proved the CLAIM for r. Let us prove it for $r + 1$.

Consider the diagram:

$$\begin{tikzcd}
\mathcal{E}_r^{p-r,q+r-1} \arrow[r, "d_r"] \arrow[d, "\phi_r'"] & \mathcal{E}_r^{pq} \arrow[r, "d_r"] \arrow[d, "\phi_r"] & \mathcal{E}_r^{p+r,q-r+1} \arrow[d, "\phi_r''"] \\
\mathcal{F}_r^{p-r,q+r-1} \arrow[r, "d_r"] & \mathcal{F}_r^{pq} \arrow[r, "d_r"] & \mathcal{F}_r^{p+r,q-r+1}
\end{tikzcd}$$

Note that if ϕ_r' and ϕ_r are isomorphisms and ϕ_r'' is monic, then ϕ_{r+1}^{pq} is iso. Since this happens by the inductive hypothesis for $p \leq r - 1$, we have isomorphisms in that same range of p.

If $-r \leq p \leq -1$, then ϕ'_r is an isomorphism and ϕ_r is monic so that, by Lemma 3.2.8, ϕ^{pq}_{r+1} is monic in this range for p and the CLAIM follows.

By taking $r \gg 0$, we get the injectivity statement for ϕ_∞.

Since $\phi^{0q}_\infty = 0$, $L^0 \subseteq \operatorname{Ker} H^*(\phi) : H_{\mathcal{E}} \to H_{\mathcal{F}}$. As to the reverse inclusion we argue by contradiction. Let $a \in \operatorname{Ker} H^*(\phi) \setminus L^0$. There exists a unique $p \leq -1$ such that $0 \neq [a] \in L^p H_{\mathcal{E}}/L^{p+1}H_{\mathcal{E}}$. One would have $\phi^{p\bullet}_\infty([a]) = 0$, violating the injectivity of the ϕ_∞ in this range for p. This proves (1).

The proof of (2) is analogous except, possibly, for the equality $L^1 H_{\mathcal{F}} = \operatorname{Im} H^*(\phi)$. Since $\phi^{0\bullet}_\infty = 0$, $L^1 H_{\mathcal{F}} \supset \operatorname{Im} H^*(\phi)$. The reverse inclusion can be proved as follows. Let $f \in L^1 H_{\mathcal{F}}$. By the surjectivity of $\phi^{1,q}_\infty$, there exists $e_1 \in L^1 H_{\mathcal{E}}$ with $(f - \phi(e_1)) \in L^2 H_{\mathcal{F}}$. We repeat this procedure using the successive surjections and conclude by induction. $\square$

Let us explain the notation in the two propositions that follow.

In the applications we have in mind, i.e. the proof of Theorem 3.3.1, the spectral sequence $\mathcal{E}_{-i}$ of Proposition 3.2.10 is a suitable realignment of the perverse spectral sequence for the cohomology group $H^*(Y_{-i}, K_{|Y_{-i}})$ of the element Y_{-i} of an n-flag. The intervals $[-n, -i]$ are explained as follows. We work with an affine variety Y of dimension n, with a complex $K \in \mathcal{D}_Y$, and we choose an n-flag with closed subsets $Y_{-i} \subseteq Y$ which are affine and of dimension $n - i$. By virtue of the Theorem 2.0.1 on the cohomological dimension of affine varieties with respect to perverse sheaves, the perverse spectral sequence $\mathcal{E}_2^{s,t}$ for the cohomology, the complex $K_{|Y_{-i}}$ is displayed on the quadrants II/III with nontrivial entries only in the columns labelled by $s \in [-(n-i), 0]$. The restriction maps induce maps of spectral sequences. But this is not what we are looking for; see Remarks 3.2.6 and 3.2.4. We want to move to the left the display of the perverse spectral sequence for Y_{-i} so that the interval of nonzero s-columns becomes $[-n, -i]$ and is thus "aligned" with the perverse spectral sequence for $H^*(Y, K)$, for which the corresponding interval is $[-n, 0]$. We want to do so and still have interesting maps of spectral sequences. This is made possible by Lemma 3.2.3. The use of the re-numerated spectral sequences, i.e. $\mathcal{E}$ instead of E (cf. §3.1.4), is not necessary, but it makes it visually easier to analyze the realigned maps of spectral sequences and deduce strong injectivity/surjectivity properties by virtue of the Lefschetz hyperplane Theorem 2.0.3 and Lemma 3.2.9.

In short, in Proposition 3.2.10, the hypothesis (a) mirrors Theorem 2.0.1 on the cohomological dimension of affine varieties and the hypothesis (b) mirrors the Lefschetz hyperplane Theorem 2.0.3.

The choice of notation in Proposition 3.2.11 is dictated by similar considerations, adapted to the case of cohomology with compact supports.

Proposition 3.2.10. *Let $n \geq 0$ be a fixed integer and*

$$\mathcal{E}_0 \longrightarrow \mathcal{E}_{-1} \longrightarrow \mathcal{E}_{-2} \longrightarrow \ldots \longrightarrow \mathcal{E}_{-n} \longrightarrow \mathcal{E}_{-n-1} := 0$$

be maps of spectral sequences. Denote by $\phi(i,j) = \phi_r^{pq}(i,j) : (\mathcal{E}_r^{pq})_{-i} \to (\mathcal{E}_r^{pq})_{-j}$ the obvious maps for $i < j$.

Assume that

(a) *$(\mathcal{E}_2^{pq})_{-i} = 0$ for $p \notin [-n, -i]$ and that*

(b) *for every $0 \leq i \leq n$, $\phi_2^{pq}(-i, -i-1)$ is an iso for $p \leq -i-2$ and monic for $p = -i-1$ (it is automatically zero for $p \geq -i$ by* (a)*).*

Then for every $\forall\, 0 \leq i \leq n$:

$$L^{-i} H_{\mathcal{E}_0} = \operatorname{Ker} \{\phi(0, -i-1) : H_{\mathcal{E}_0} \longrightarrow H_{\mathcal{E}_{-i-1}}\}.$$

Proof. The case $n = 0$ is trivial. The system of $\mathcal{E}_{-i}$ with $i > 0$ can be made to satisfy the hypothesis of this proposition with $n-1$ in place of n by shifting the display one unit to the right.

The proof is by induction on n.

The case $i = 0$ is covered by Lemma 3.2.9, so we may assume that $i > 0$.

Assume we have proved the proposition for $n-1$ and let us prove it for n. The induction hypothesis translates into

$$L^{-i} H_{\mathcal{E}_{-1}} = \operatorname{Ker} \{\phi(-1, -i-1) : H_{\mathcal{E}_{-1}} \longrightarrow H_{\mathcal{E}_{-i-1}}\}, \qquad \forall\, 1 \leq i \leq n.$$

Since $L^{-i} H_{\mathcal{E}_0} \to L^{-i} H_{\mathcal{E}_{-1}}$, we see that

$$L^{-i} H_{\mathcal{E}_0} \subseteq \operatorname{Ker} \{\phi(0, -i-1) : H_{\mathcal{E}_0} \longrightarrow H_{\mathcal{E}_{-i-1}}\}, \qquad \forall\, 1 \leq i \leq n$$

and we already know the case $i = 0$.

To prove the reversed inclusion, we argue by contradiction. Let $a \in \operatorname{Ker} \phi(0, -i-1)$. Note that its image $a' \in H_{\mathcal{E}_{-1}}$ maps automatically to zero into $H_{\mathcal{E}_{-i-1}}$ so that, by the inductive hypothesis, $a' \in L^{-i} H_{\mathcal{E}_{-1}}$.

Assume that $a \notin L^{-i} H_{\mathcal{E}_0}$. Then $\exists!\, s > i$ such that $a \in L^{-s} H_{\mathcal{E}_0} \setminus L^{-s+1} H_{\mathcal{E}_0}$, so that $0 \neq [a]_{-s} \in (\mathcal{E}_\infty^{-s, \bullet})_0$.

By Lemma 3.2.9(1), the image of this element, $[a']_q$ in $(\mathcal{E}_\infty^{-s, \bullet})_{-1}$ is not zero. This would imply that $a' \notin L^{-s+1} H_{\mathcal{E}_{-1}} \supseteq L^{-i} H_{\mathcal{E}_{-1}}$; a contradiction. $\square$

Analogously, we have the following proposition.

Proposition 3.2.11. *Let $n \geq 0$ be a fixed integer and*

$$0 =: \mathcal{E}_{n+1} \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0$$

be maps of spectral sequences. Denote by $\phi(i,j) : \mathcal{E}_i \to \mathcal{E}_j$ the obvious maps for $i \geq j$.

Assume that

(a) $(\mathcal{E}_2^{pq})_i = 0$ *for* $p \notin [i, n]$ *and that*

(b) *for every* $0 \leq i \leq n$, $\phi_2^{pq}(i, i-1)$ *is an iso for* $p \geq i+1$ *and epic for* $p = i$ *(it is automatically zero for* $p \leq i-1$ *by* (a)*).*

Then

$$L^i H_{\mathcal{E}_0} = \mathrm{Im}\,\{\phi(i,0) : H_{\mathcal{E}_i} \longrightarrow H_{\mathcal{E}_0}\}, \qquad \forall\, 0 \leq i \leq n.$$

Proof. Completely analogous to the proof of Proposition 3.2.10, via Lemma 3.2.9(2). □

3.2.3. Jouanolou Trick: reduction to the affine case. In this section, we prove Lemma 3.2.12 which is one way to reduce the study of the perverse spectral sequence on a quasi-projective variety Y, to the case of affine varieties by replacing Y with the affine $\mathcal{Y}$. For a different approach, using two flags, see [11].

For the notions of translated spectral sequence $\mathcal{E}(l)$ with filtration $L(l)$ see §3.1.4.

Let Y be a quasi-projective variety of dimension n, let $f : X \to Y$ be a map and let $K \in \mathcal{D}_Y$, $C \in D_X$.

By fixing an $\mathbb{A}^d$-fibration $\pi : \mathcal{Y} \to Y$ as in Proposition 2.0.6, we obtain the Cartesian diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi"] \arrow[d, "f"] & X \arrow[d, "f"] \\ \mathcal{Y} \arrow[r, "\pi"] & Y. \end{tikzcd} \tag{3.21}$$

Since the fibers of the maps π are affine spaces, we have canonical identifications

$$\begin{aligned} Id \simeq \pi_*\pi^*, &\quad H^*(Y, K) = H^*(\mathcal{Y}, \pi^*K), \\ \pi_!\pi^! \simeq Id, &\quad H_c^*(\mathcal{Y}, \pi^!K) = H_c^*(Y, K); \end{aligned} \tag{3.22}$$

in fact, the first one follows from [19], Corollary 2.7.7(ii), the second follows from the first one, the fourth from the third and the third from the first one in view of the fact that Poincaré-Verdier duality exchanges the pull-back π^* with the extraordinary pull-back $\pi^!$.

Since π is smooth of relative dimension d, we have a canonical identification of functors

$$\pi^*[d] = \pi^![-d]. \tag{3.23}$$

The functors $\pi^*[d] = \pi^![-d]$ are t-exact with respect to middle perversity. The functors $\pi^* = \pi^![-2d]$ are exact in the usual sense.

Given a flag $\mathcal{Y}_*$ on $\mathcal{Y}$, we have the pre-image flag $\mathcal{X}_* := f^{-1}\mathcal{Y}_*$ on $\mathcal{X}$. Recall that we have the two associated filtrations $F_{\mathcal{Y}_*}$ and $G_{\mathcal{Y}_*}$ of §3.1.6.

Lemma 3.2.12. *There are canonical identifications of filtered abelian groups:*

$$H^*(Y,K) = H^*(\mathcal{Y},\pi^*K), \qquad L^Y_{\mathfrak{p}_\tau} = L^{\mathcal{Y}}_{\mathfrak{p}_\tau}(-d), \qquad L^Y_\tau = L^{\mathcal{Y}}_\tau, \tag{3.24}$$

$$H_c(\mathcal{Y},\pi^!K) = H_c(Y,K), \qquad L^Y_{\mathfrak{p}_\tau} = L^{\mathcal{Y}}_{\mathfrak{p}_\tau}(d), \qquad L^Y_\tau = L^{\mathcal{Y}}_\tau(2d). \tag{3.25}$$

We have the following relations in $H^*(X,C)$*:*

$$F_{\mathcal{X}_*} \supseteq F_{\mathcal{Y}_*}, \qquad L^{f:X\to Y}_{\mathfrak{p}_\tau} = L^{f:\mathcal{X}\to\mathcal{Y}}_{\mathfrak{p}_\tau}(-d), \qquad L^{f:X\to Y}_\tau = L^{f:\mathcal{X}\to\mathcal{Y}}_\tau, \tag{3.26}$$

and the following relations in $H^*_c(X,C)$:

$$G_{\mathcal{X}_*} \subseteq G_{\mathcal{Y}_*}, \qquad L^{f:\mathcal{X}\to\mathcal{Y}}_{\mathfrak{p}_\tau}(d) = L^{f:X\to Y}_{\mathfrak{p}_\tau}, \qquad L^{f:X\to Y}_\tau = L^{f:\mathcal{X}\to\mathcal{Y}}_\tau(2d). \tag{3.27}$$

Proof. The statements (3.24) and (3.25) follow from (3.22), the t-exactness of $\pi^*[d] = \pi^![-d]$, the exactness of $\pi^* = \pi^![-2d]$ and from Lemma 3.2.3.

The inclusion in (3.26) is seen as follows. There is the commutative diagram,

$$\begin{tikzcd}
H^*(\mathcal{X},\pi^*C) \arrow[d, "a"] & = & & H^*(\mathcal{Y}, f_*\pi^*C) \arrow[d, "a'"] \\
H^*(\mathcal{X}_s, i_s^*\pi^*C) & = & H^*(\mathcal{Y}_s, f_*i_s^*\pi^*C) & H(\mathcal{Y}_s, i_s^*f_*\pi^*C), \arrow[l, "b"']
\end{tikzcd} \tag{3.28}$$

where b stems from the base change map (2.4) $i_s^*f_* \to f_*i_s^*$. The kernels of the vertical restriction maps a and a' define the filtrations $F_{\mathcal{X}}$ and $F_{\mathcal{Y}}$ and it is clear that $\operatorname{Ker} a \supseteq \operatorname{Ker} a'$.

The second (third, respectively) equality in (3.26) follows from the definition of $L^f_{\mathfrak{p}_\tau}$ (of L^f_τ, respectively) the smooth base change isomorphism $f_*\pi^* = \pi^*f_*$ for the smooth map π and the second (third, respectively) equality in (3.24).

The proof of (3.27) runs parallel to the one just given via the use of the base change map $f_!i_s^! \to i_s^!f_!$ and the base change isomorphism $f_!\pi^! = \pi^!f_!$. The inclusion is reversed with respect to the one in cohomology, and this is because the flag filtration in cohomology with compact supports is, by definition, given by the images of the co-restriction maps. □

Remark 3.2.13. The proof of Lemma 3.2.12 makes it clear that the failure of the base change map to be an isomorphism is responsible for the inequality $F_{\mathcal{X}_*} \neq F_{\mathcal{Y}_*}$. If $f : X \to Y$ is proper, then, the base change isomorphism for proper maps yields the equality $F_{\mathcal{X}_*} = F_{\mathcal{Y}_*}$ in cohomology, as well as in cohomology with compact supports. If f is not proper, then one can still have the equality $F_{\mathcal{X}_*} = F_{\mathcal{Y}_*}$ in cohomology, provided that, for every element $i_s : \mathcal{Y}_s \to \mathcal{Y}$ of the flag, the base change map $i_s^* f_* \to f_* i_s^*$ is an isomorphism when evaluated on $\pi^* C$. Similarly, in the case of cohomology with compact supports. As we show in [12], and also in [11], this can be achieved by choosing a system of general hyperplane sections. In particular, the following holds.

Proposition 3.2.14. *In the situation of* Lemma 3.2.12*, if the flag $\mathcal{Y}_*$ is chosen to be general, then we have the equality $F_{\mathcal{X}_*} = F_{\mathcal{Y}_*}$.*

3.3. The geometry of the perverse and perverse Leray filtrations. Let Y be a quasi-projective variety, let $K \in \mathcal{D}_Y$, let $f : X \to Y$ be a map, and let $C \in \mathcal{D}_X$.

Recall that, the perverse Leray filtration on $H^*(X,C)$ is defined to be the perverse filtration on $H^*(Y, f_* C)$. Similarly, for $H^*_c(X,C) = H^*_c(Y, f_! C)$.

In this section, we employ the set-up of §3.2.3 and we identify:

- the perverse filtrations on $H^*(Y,K)$ and on $H^*_c(Y,K)$ with suitable flag filtrations on the auxiliary affine variety $\mathcal{Y}$, and
- the perverse Leray filtrations on $H^*(X,C)$ and on $H^*_c(X,C)$ with suitable flag filtrations on the auxiliary variety $\mathcal{X}$.

Lemma 3.2.12, relates the perverse Leray filtration on $H^*(X,C)$ with the one on $H^*(\mathcal{X}, \pi^* C)$ and similarly for compact supports. We employ the following identifications

$$H^*(X,C) = H^*(Y, f_* C) = H^*(\mathcal{Y}, \pi^* f_* C = f_* \pi^* C) = H^*(\mathcal{X}, \pi^* C),$$

$$H^*_c(X,C) = H^*_c(Y, f_! C) = H^*_c(\mathcal{Y}, \pi^! f_! C = f_! \pi^! C) = H^*(\mathcal{X}, \pi^! C),$$

of cohomology groups and the following ones for the filtrations on them

$$L^{f:X\to Y}_{\mathfrak{p}_\mathcal{T}} := L^Y_{\mathfrak{p}_\mathcal{T}} \overset{(3.24)}{=} L^{\mathcal{Y}}_{\mathfrak{p}_\mathcal{T}}(-d) =: L^{f:\mathcal{X}\to\mathcal{Y}}_{\mathfrak{p}_\mathcal{T}}(-d), \quad \text{on } H^*(X,C), \tag{3.29}$$

$$L^{f:X\to Y}_{\mathfrak{p}_\mathcal{T}} := L^Y_{\mathfrak{p}_\mathcal{T}} \overset{(3.25)}{=} L^{\mathcal{Y}}_{\mathfrak{p}_\mathcal{T}}(d) =: L^{f:\mathcal{X}\to\mathcal{Y}}_{\mathfrak{p}_\mathcal{T}}(d), \quad \text{on } H^*_c(X,C). \tag{3.30}$$

3.3.1. The perverse filtrations on $H^*(Y,K)$ and $H^*_c(Y,K)$.

Theorem 3.3.1. *Let Y be a quasi-projective variety of dimension n and $K \in \mathcal{D}_Y$. Let $\pi : \mathcal{Y} \to Y$ be an $\mathbb{A}^d$-fibration with $\mathcal{Y}$ affine. There exists a $(n+d)$-flag $\mathcal{Y}_*$ on $\mathcal{Y}$ such that we have an equality of filtered abelian groups*

$$H^*(Y,K) = H^*(\mathcal{Y}, \pi^* K), \qquad L^Y_{\mathfrak{p}_\mathcal{T}} = F_{\mathcal{Y}_*}(-d), \tag{3.31}$$

$$H^*_c(Y,K) = H^*_c(\mathcal{Y}, \pi^! K), \qquad L^Y_{\mathfrak{p}_\mathcal{T}} = G_{\mathcal{Y}_*}(d). \tag{3.32}$$

If Y is an affine variety, then one may take $\mathcal{Y} = Y$ and $d = 0$, and then

$$L^Y_{\mathfrak{p}_\mathcal{T}} = F_{Y_*} \ \text{ on } \ H^*(Y,K), \qquad L^Y_{\mathfrak{p}_\mathcal{T}} = G_{Y_*} \ \text{ on } \ H^*_c(Y,K). \tag{3.33}$$

Proof. The proofs for cohomology and for cohomology with compact supports run parallel. By virtue of the equalities $L^Y_{{}^{\mathfrak{p}}\tau} = L^{\mathcal{Y}}_{{}^{\mathfrak{p}}\tau}(-d)$ (3.24) and $L^Y_{{}^{\mathfrak{p}}\tau} = L^{\mathcal{Y}}_{{}^{\mathfrak{p}}\tau}(d)$ (3.25) in Lemma 3.2.12, we are left with showing that we can choose $\mathcal{Y}_*$ on $\mathcal{Y}$ so that $L^{\mathcal{Y}}_{{}^{\mathfrak{p}}\tau} = F_{\mathcal{Y}_*}$ in cohomology, and $L^{\mathcal{Y}}_{{}^{\mathfrak{p}}\tau} = G_{\mathcal{Y}_*}$ in cohomology with compact supports. In particular, we may assume that $\mathcal{Y} = Y$ is affine.

Let Σ be a stratification of Y with the property that K is Σ-constructible, i.e. $K \in \mathcal{D}^{\Sigma}_{Y}$.

Let $i : Y_{-1} \to Y$ be a general hyperplane section of Y. Then, for every $m \in \mathbb{Z}$, the complexes $i^* \, {}^{\mathfrak{p}}\mathcal{H}^m(K)[-1] = i^! \, {}^{\mathfrak{p}}\mathcal{H}^m(K)[1] \in \mathcal{P}_{Y_{-1}}$, i.e. they are perverse on Y_{-1}.

By Theorem 2.0.1 on the cohomological dimension of affine varieties and by the Lefschetz hyperplane Theorem 2.0.3, there is a general (cf. Remark 2.0.2) hyperplane section $i := i_{-1} : Y_{-1} \to Y$ such that, for every $m \in \mathbb{Z}$, the natural maps

$$a^j : H^j(Y, {}^{\mathfrak{p}}\mathcal{H}^m(K)) \longrightarrow H^j(Y_{-1}, i^* \, {}^{\mathfrak{p}}\mathcal{H}^m(K)),$$

$$b^j : H^l_c(Y_{-1}, i^! \, {}^{\mathfrak{p}}\mathcal{H}^m(K)) \longrightarrow H^l_c(Y, {}^{\mathfrak{p}}\mathcal{H}^m(K)),$$

satisfy the following conditions:

$$\begin{aligned} H^j(Y, {}^{\mathfrak{p}}\mathcal{H}^m(K)) = 0, &\quad j \notin [-n, 0], \\ H^j(Y_{-1}, i^* \, {}^{\mathfrak{p}}\mathcal{H}^m(K)) = 0, &\quad j \notin [-n, -1], \end{aligned} \tag{3.34}$$

$$a^j \text{ is iso for } j \in [-n, -2], \qquad a^{-1} \text{ is monic;} \tag{3.35}$$

$$\begin{aligned} H^j_c(Y, {}^{\mathfrak{p}}\mathcal{H}^m(K)) = 0, &\quad j \notin [0, n], \\ H^j_c(Y, i^! \, {}^{\mathfrak{p}}\mathcal{H}^m(K)) = 0, &\quad j \notin [1, n], \end{aligned} \tag{3.36}$$

$$b^j \text{ is iso for } j \in [2, n], \qquad b^1 \text{ is epic.} \tag{3.37}$$

Note that the complex i^*K is constructible with respect to the stratification Σ_{-1} on Y_{-1} induced by Σ and that $i^*[-1] = i^![1] : \mathcal{D}^{\Sigma}_{Y} \to \mathcal{D}^{\Sigma_{-1}}_{Y_{-1}}$ are t-exact.

We iterate the construction and take Y_{-l-1} to be a general hyperplane section of Y_{-l}.

Let $\mathcal{E}_{-l}$ be the perverse spectral sequence (3.12) for $H^*(Y_{-l}, K_{|Y_{-l}})$ translated by $+l$:

$$\mathcal{E}_{-l} := \mathcal{E}(K_{|Y_{-l}}, {}^{\mathfrak{p}}\tau)(+l). \tag{3.38}$$

Let ${}_c\mathcal{E}_l$ be the perverse spectral sequence (3.12) for $H_c(Y_{-l}, i^!_{-l}K)$ translated by $-l$:

$${}_c\mathcal{E}_l := {}_c\mathcal{E}(i^!_{-l}K), {}^{\mathfrak{p}}\tau)(-l).$$

Since $i^*_{-l}[-l] = i^!_{-l}[l]$ are t-exact, Lemma 3.2.3 yields two systems of maps of spectral sequences

$$\mathcal{E}_0 \longrightarrow \mathcal{E}_{-1} \longrightarrow \ldots \longrightarrow \mathcal{E}_{-n} \longrightarrow \mathcal{E}_{-n-1} = 0,$$
$$0 = \mathcal{E}_{n+1} \longrightarrow \mathcal{E}_n \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0.$$

By (3.34) and (3.35), this system of spectral sequences satisfies the hypotheses of Proposition 3.2.10 and the conclusion for cohomology follows.

By (3.36) and (3.37), this system of spectral sequences satisfies the hypotheses of Proposition 3.2.11 and the conclusion for cohomology with compact supports follows. □

Remark 3.3.2. Let us write out the conclusions above when Y is affine:

$$L^p_{\mathfrak{p}_\mathcal{T}}\, H^j(Y,K) = \mathrm{Ker}\{H^j(Y,K) \longrightarrow H^j(Y_{p-1}, K_{|Y_{p-1}}\}, \qquad \forall\, p \in \mathbb{Z},$$
$$L^p_{\mathfrak{p}_\mathcal{T}}\, H^j_c(Y,K) = \mathrm{Im},\{H^j_c(Y_p, i^!_p K) \longrightarrow H^j_c(Y,K)\}, \qquad \forall\, p \in \mathbb{Z}.$$

Remark 3.3.3. As the proof shows, one can choose the flag $\mathcal{Y}_*$ by taking $(n+d)$ general hyperplane sections of $\mathcal{Y}$. In fact, one can take $(n+d)$ general hypersurface sections of $\mathcal{Y}$ of varying degrees. The subspaces of the filtrations can be viewed, in cohomology, as kernels of the restriction maps and, in cohomology with compact supports, as images of the co-restriction maps associated with the maps $\pi_l := \pi \circ i_l : \mathcal{Y}_l \to Y$.

Remark 3.3.4 (Type of perverse filtrations)**.** By trivial reasons of indexing, the flag filtration F_{Y_*} is of type $[-n, 0]$ on each group $H^*(Y,K)$, and Theorem 2.0.1 on the cohomological dimension of affine varieties implies that if Y is affine, then the perverse filtration $L_{\mathfrak{p}_\mathcal{T}}$ on the group $H^*(Y,K)$ is of type $[-n, 0]$. This is in accordance with Theorem 3.3.1, i.e. if Y is affine of dimension n and $Y_* \subseteq Y$ is a general n-flag of linear sections, then $L_{\mathfrak{p}_\mathcal{T}} = F_{Y_*}$. Similarly, for cohomology with compact supports, where the type of G_{Y_*} on the affine Y is $[0, n]$. If Y is quasi-projective, then Theorem 3.3.1 yields $[-n, d]$ and $[-d, n]$ as bounds on the type of the perverse filtrations $L_{\mathfrak{p}_\mathcal{T}}$ in cohomology and in cohomology with compact supports, respectively, where d is the dimension of the fiber $\mathbb{A}^d$ of the chosen map $\pi : \mathcal{Y} \to Y$. These bounds on $L^Y_{\mathfrak{p}_\mathcal{T}}$ are *not* sharp. One needs to replace d by t, where $t+1$ is the smallest number of affine open sets for an open covering of Y given by affine open subvarieties. However, this does not create problems in applications.

3.3.2. The perverse Leray filtrations on $H^*(X,C)$ and $H^*_c(X,C)$. We are aiming at a geometric description of the perverse Leray filtrations using a flag on $\mathcal{X}$. The obvious candidate is the flag $\mathcal{X}_* := f^{-1}\mathcal{Y}_*$.

Recall that the inclusions (3.26) and (3.27) in Lemma 3.2.12 can be strict in view of the possible failure of the relevant base change of a general flag $\mathcal{Y}_*$ on $\mathcal{Y}$ corrects this failure.

Theorem 3.3.5. *Let $f : X \to Y$ be a map of varieties where Y is quasi-projective of dimension n. Let $\pi : \mathcal{Y} \to Y$ be an $\mathbb{A}^d$-fibration with $\mathcal{Y}$ affine and let $\mathcal{X} = \mathcal{Y} \times_Y X$. Then there is a $(n+d)$-flag $\mathcal{X}_*$ on $\mathcal{X}$ such that*

$$L^{f:X\to Y}_{{}^{\mathfrak{p}}\tau} = F_{\mathcal{X}_*}(-d) \;\; \text{on } H^*(X,C), \quad \text{and} \quad L^{f:X\to Y}_{{}^{\mathfrak{p}}\tau} = G_{\mathcal{X}_*}(d) \;\; \text{on } H^*_c(X,C).$$

If Y is an affine variety, then we may take $\mathcal{X} = X$ and $d = 0$ and then

$$L^{f:X\to Y}_{{}^{\mathfrak{p}}\tau} = F_{X_*} \;\; \text{on } H^*(X,C), \quad \text{and} \quad L^{f:X\to Y}_{{}^{\mathfrak{p}}\tau} = G_{X_*} \;\; \text{on } H^*_c(X,C).$$

Proof. We prove the result for cohomology. The case of compact supports is analogous. Choose a $(n+d)$-flag $\mathcal{Y}_*$ on $\mathcal{Y}$ using $(n+d)$ general hyperplane sections of $\mathcal{Y}$ such that, having set $\mathcal{X}_* := f^{-1}\mathcal{Y}_*$, we have:

$$L^{\mathcal{Y}}_{{}^{\mathfrak{p}}\tau} \overset{Thm.\ 3.3.1}{=} F_{\mathcal{Y}_*} \overset{Prop.\ 3.2.14}{=} F_{\mathcal{X}_*}.$$

We conclude by using the equality (3.29) $L^{\mathcal{Y}}_{{}^{\mathfrak{p}}\tau}(-d) = L^{f:X\to Y}_{{}^{\mathfrak{p}}\tau}$. □

Remark 3.3.6. If Y is affine, then we have

$$(L^f_{{}^{\mathfrak{p}}\tau})^p H^j(X,C) = \mathrm{Ker}\{H^*(X,C) \longrightarrow H^j(X_{p-1}, C_{|X_{p-1}})\},$$
$$(L^f_{{}^{\mathfrak{p}}\tau})^p H^j_c(X,C) = \mathrm{Im}\,\{H^j_c(X_p, i^!_p C) \longrightarrow H^j_c(X,C)\}.$$

Remark 3.3.7. Note that in Theorem 3.3.5 we do not need to assume that X is quasi-projective, nor that f is proper.

Remark 3.3.8. D. Arapura [1] proved that if $f : X \to Y$ is a projective map of quasi-projective varieties, then the *standard* Leray filtration on the cohomology groups $H^*(X,\mathbb{Z})$ can be described geometrically using flags in *special* position. His methods are different from the ones of the present paper (and of [11]). However, I feel a great intellectual debt to [1]. I do not know of a way to describe the standard Leray filtration in cohomology using flags on (varieties associated with) the domain of a nonproper map. The paper [12] is devoted to provide such a description for the Leray filtration on cohomology with compact supports via compactifications of varieties and of maps.

4. Applications of the results on filtrations

Recall that the acronym MH(S)S stands for mixed Hodge (sub)structure. Due to the functorial nature of the canonical MHS on algebraic varieties, the description of perverse filtrations in terms of flags, i.e. as kernel of restrictions and as images of co-restrictions to subvarieties, is amenable to applications to the mixed Hodge theory of the cohomology and intersection cohomology of quasi-projective varieties. In this section we work out some of these applications.

4.1. Perverse Leray and MHS: singular cohomology. The following theorem appears in a stronger form (involving filtered complexes and spectral sequences) in [11] and we include it here for the reader's convenience as it is an important preliminary result to the applications that follow.

Theorem 4.1.1. *Let $f : X \to Y$ be a map of algebraic varieties with Y quasi-projective. There is an integer d, a variety $\mathcal{X}$, and a flag $\mathcal{X}_*$ on $\mathcal{X}$ such that there are identities of filtered groups*

$$(H^*(X,\mathbb{Z}), L^f_{\mathfrak{p}_\tau}) \qquad = \qquad (H^*(\mathcal{X},\mathbb{Z}), F_{\mathcal{X}_*}(-d)),$$

$$(H^*_c(X,\mathbb{Z}), L^f_{\mathfrak{p}_\tau}) \qquad = \qquad (H^*_c(\mathcal{X},\mathbb{Z}), G_{\mathcal{X}_*}(d)).$$

In particular, the perverse Leray filtrations on $H^*(X,\mathbb{Z})$ and $H^*_c(X,\mathbb{Z})$ are by MHSS.

Proof. The first statement is a mere application of Theorem 3.3.5 to the case $C = \mathbb{Z}_X$, and the subspaces of the perverse filtrations are the kernels of the restriction maps $H^*(X,\mathbb{Z}) = H^*(\mathcal{X},\mathbb{Z}) \to H^*(\mathcal{X}_p,\mathbb{Z})$ and the images of the co-restriction maps $H_c^{*+2d+2p}(\mathcal{X}_p,\mathbb{Z}) \to H_c^{*+2d}(\mathcal{X},\mathbb{Z}) = H^*_c(X,\mathbb{Z})$, respectively. The second statement about MHSS follows from the usual functoriality properties of the MHS on the cohomology groups and on the cohomology groups with compact supports of varieties [15]. □

Remark 4.1.2. The case of the Leray filtration in cohomology is dealt with in [12] where the relation with Arapura's results [1] is discussed.

Remark 4.1.3. Theorem 4.1.1 holds if we replace cohomology with intersection cohomology; see Proposition 4.5.4. The proof is formally analogous. However, we must first endow intersection cohomology with an MHS (Theorem 4.3.1) and then verify the compatibility of the resulting MHS with restrictions to general hyperplane sections (see the proof of Proposition 4.5.4).

4.2. Review of the decomposition theorem. In the sequel of this paper, we employ $\mathbb{Q}$-coefficients (we still denote the corresponding category $\mathcal{D}_Y$). The reason for employing rational coefficients stems from the use of the decomposition theorem, which does not hold over the integers. Moreover, we use Poincaré-Verdier duality in the form of perfect pairings between rational vector spaces.

We also assume that X and Y are irreducible and quasi-projective of dimension m and n, respectively. This only because it makes the exposition simpler. All the results we prove hold without the irreducibility assumption, with pretty much the same proofs.

Let $f : X \to Y$ be a proper map. The decomposition theorem, due to Beilinson-Bernstein-Deligne-Gabber (see the survey [13]) yields the existence of direct sum decompositions in $\mathcal{D}_Y$ for the direct image of the (rational) intersection complex of X:

$$\phi: \; f_* IC_X \; \simeq \; \bigoplus_{i,l,S} IC_{\overline{S}}(L_{i,l,S})[-i], \tag{4.1}$$

where $i \in \mathbb{Z}$, $0 \le l \le n$, $Y = \coprod_{0 \le l \le n} S_l$ is a stratification of Y, part of a stratification for the map f, S_l is the nonnecessarily connected l-dimensional stratum, S ranges over the set of connected components of S_l, and for every i, l, S, the symbol $L_{i,l,S}$ denotes a semisimple local system of rational vector spaces on S.

By grouping the summands with the same shift $[-i]$, we obtain canonical identifications

$$^{\mathfrak{p}}\mathcal{H}^i(f_* IC_X) = \bigoplus_{l,S} IC_{\overline{S}}(L_{i,l,S}). \tag{4.2}$$

Recall that $IH^j(X) = H^{j-m}(X, IC_X)$ and that $IH^j_c(X) = H^{j-m}_c(X, IC_X)$.

We define the perverse cohomology groups of X (with respect to f) by setting

$$\begin{aligned} IH^j_i(X) &:= H^{j-i-m}(Y, {}^{\mathfrak{p}}\mathcal{H}^i(f_* IC_X)), \\ IH^j_{i,l,S}(X) &:= \bigoplus_{l,S} H^{j-i-m}(Y, IC_{\overline{S}}(L_{i,l,S})). \end{aligned} \tag{4.3}$$

While there may be no natural choice for the isomorphism ϕ in (4.1), the perverse cohomology groups are natural subquotients of the groups $IH^*(X)$.

We define the perverse cohomology groups with compact supports in a similar way.

We have canonical identifications

$$IH^j_i(X) = \bigoplus_{l,S} IH^j_{i,l,S}(X), \qquad IH^j_{c,i}(X) = \bigoplus_{l,S} IH^j_{c,i,l,S}(X). \tag{4.4}$$

Recall that if X is nonsingular, then $IC_X = \mathbb{Q}_X[m]$ and $IH^j(X) = H^j(X)$, etc. In this case, we denote the perverse cohomology groups as follows:

$$H^j_i(X), \qquad H^j_{c,i}(X), \qquad H^j_{i,l,S}(X), \qquad H^j_{c,i,l,S}(X). \tag{4.5}$$

4.3. Decomposition theorem and mixed Hodge structures. In this section we state some mixed-Hodge-theoretic results concerning the intersection cohomology of quasi-projective varieties. These results are proved in §§4.4 and 4.5 by making use of the geometric description of the perverse Leray filtration given in Theorem 4.1.1.

Theorem 4.3.1 (MHS on intersection cohomology). *Let Y be an irreducible quasi-projective variety of dimension n. Then*

(1) *The groups $IH^*(Y)$ and $IH^*_c(Y)$ carry a canonical MHS. If Y is nonsingular, then this MHS coincides with Deligne's* [15]. *If $f : X \to Y$ is a resolution of the singularities of Y, then the MHS on $IH^*(Y)$ and $IH^*_c(Y)$ are canonical subquotient MHS of the MHS on $H^*(X)$ and $H^*_c(X)$, respectively.*

(2) *The Goresky-MacPherson-Poincaré duality isomorphism yields isomorphism of MHS*

$$IH^j(Y) \simeq IH_c^{2n-j}(Y)^\vee(-n).$$

(3) *The canonical maps $a : H^*(Y) \to IH^*(Y)$, $a' : H^*_c(Y) \to IH^*_c(Y)$ are maps of MHS. If Y is projective, then* $\mathrm{Ker}\,\{a : H^j(Y) \to IH^j(Y)\} = W_{j-1}H^j(Y)$ *(the subspace of weights $\leq j-1$).*

Theorem 4.3.2 (Pieces of the decomposition theorem and MHS). *Let $f : X \to Y$ be a projective map of quasi-projective irreducible varieties and let $m := \dim X$. Then:*

(1) *The subspaces of the perverse Leray filtrations on $IH^*(X)$ and on $IH^*_c(X)$ are MHSS for the MHS of Theorem* 4.3.1.

(2) *The perverse cohomology groups $IH^j_i(X)$ and $IH^j_{c,i}(X)$ in* (4.3) *carry natural MHS which are subquotients of the natural MHS on $IH^j(X)$ and on $IH^j_c(X)$, respectively.*

(3) *The subspaces $IH^j_{i,l,S}(X) \subseteq IH^j_i(X)$ and $IH^j_{c,i,lS}(X) \subseteq IH^j_{c,i}(X)$ in* (4.4) *are MHSS.*

(4) *The Poincaré pairing isomorphisms of Theorem* 4.3.1 *applied to X descend to the perverse cohomology groups and induce isomorphisms of MHS*

$$IH^j_i(X) \simeq IH^{2m-j}_{c,-i}(X)^\vee(-m), \qquad IH^j_{i,l,S}(X) \simeq IH^{2m-j}_{c,-i,l,S}(X)^\vee(-m).$$

Theorem 4.3.3 (Hodge theoretic splitting ϕ). *There exist splittings ϕ in* (4.1) *for which the splittings*

$$\phi \,:\, IH^j(X,\mathbb{Q}) \simeq \bigoplus_{i,l,S} IH^j_{i,l,S}(X), \qquad \phi \,:\, IH^j_c(X,\mathbb{Q}) \simeq \bigoplus_{c,i,l,S} IH^j_{c,i,l,S}(X)$$

are isomorphisms of MHS for the MHS in Theorem 4.3.1, *part* (1), *and Theorem* 4.3.2, *part* (3).

Theorem 4.3.4 (Induced morphisms in intersection cohomology). *Let $f : X \to Y$ be a proper map of quasi-projective irreducible varieties. There is a canonical splitting injection*

$$\gamma : IC_Y \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^{\dim Y - \dim X}(f_* IC_X)$$

and there is a choice of ϕ *in* (4.1) *that yields a commutative diagram of MHS:*

$$\begin{tikzcd} H^j(Y) \arrow[r, "a_Y"] \arrow[d, "f^*"] & IH^j(Y) \arrow[d, "\phi\gamma"] \\ H^j(X) \arrow[r, "a_X"] & IH^j(X). \end{tikzcd}$$

Saito's work [21, 22] on mixed Hodge modules (MHM) implies all the results on MHS we prove in §4, with one caveat: it is not a priori clear that the MHS coming from the theory of MHM coincide with the ones of this paper.

Theorem 4.3.5 (Comparison with M. Saito's MHS)**.** *The MHS appearing in Theorems* 4.3.1 *and* 4.3.2 *coincide with the ones arising from MHM.*

4.4. Scheme of proof of the results in §4.3. With the exception of Theorem 4.3.5, all the results listed in §4.3 were proved in [8, 10] in the case when X and Y are projective varieties (in which case all the Hodge structures in question are pure).

Roughly speaking, we first prove the results in 4.3 in the special case when X is nonsingular, and then we use resolution of singularities to conclude.

Let me outline more precisely the structure of the proofs. In fact, in what follows we prove these results, except for certain assertions which are then proved in §4.5, Propositions 4.5.1, 4.5.4 and Lemmata 4.5.2, 4.5.3. Many of the details carry over verbatim from the projective case and will not be repeated here; we simply point the reader to the original proofs in [8, 10]. Some other details which seem to be less routine are spelled out.

(1) We prove Theorem 4.3.2, parts (1), (2) and (4) in the case when X is nonsingular by using the geometric description of the perverse Leray filtration given by Theorem 4.1.1. This is done in Proposition 4.5.1.

(2) We prove Theorem 4.3.2, part (3) in the case when X is singular in Lemmata 4.5.2 and 4.5.3. These two lemmata are adapted from [8], Lemma 7.1.1 and proof of the purity Theorem 2.2.1.

Theorem 4.3.2 is thus proved in the case when X is nonsingular. In order to tackle the case when X is singular, we must first endow intersection cohomology groups with an MHS, i.e. we must now prove Theorem 4.3.1.

(3) Proof of Theorem 4.3.1. The projective case is proved in [8]. We follow the same strategy and point out the needed modifications. Let $f : X \to Y$ be a resolution of the singularities of Y. By the decomposition theorem, the intersection cohomology groups $IH^*(Y)$ and $IH^*_c(Y)$ are the subspaces of the quotient perverse cohomology groups $H^*_0(X)$

and $H_c^*(X)$, respectively, that correspond to the unique dense stratum on Y. Part (1) follows by applying Theorem 4.3.2, part (3), which we have proved in the case when X is nonsingular. The MHS so-obtained are shown to be independent of the resolution by an argument identical to the one in the proof of [8], proof of Theorem 2.2.3.a. The proof of part (2) follows from Theorem 4.3.2, part (4) applied to f (X is nonsingular) when we consider the dense stratum on Y. The proof of part (3) is identical to the one for the projective case.

We can now complete the proof of Theorem 4.3.2 by dealing with the case when X is singular.

(4) Theorem 4.3.2, part (1) follows directly from Proposition 4.5.4. This proposition is the intersection cohomology analogue of Theorem 4.1.1 and it is proved in pretty much the same way. We only need to verify that by taking general linear sections, the restriction maps on the intersection cohomology groups, and the co-restriction maps on the intersection cohomology groups with compact supports are compatible with the MHS of Theorem 4.3.1.

(5) Theorem 4.3.2, part (1) clearly implies Theorem 4.3.2, part (2).

(6) Proof of Theorem 4.3.2, part (3). We discuss the case of cohomology. The case of cohomology with compact supports is analogous. Let $g : X' \to X$ be a resolution of the singularities of X. Set $h := f \circ g : X' \to Y$. Since X' is nonsingular, Theorem 4.3.2 holds for g and for h. Let F^g_a and F^h_b denote the increasing perverse Leray filtrations on the groups $H^*(X')$ associated with the maps g and h, respectively. Let $Gr^{F^g}_a H^*(X')$, $Gr^{F^h}_b H^*(X')$ denote the corresponding graded pieces. By Theorem 4.3.2, part (1), the subspaces of both filtrations are MHSS. The graded pieces, as well as all the bi-graded pieces $Gr^{F^g}_a Gr^{F^h}_b H^*(X')$ inherit the natural subquotient MHS. The same is true, via the just-established Theorem 4.3.2, part (1), for the groups $IH^*(X)$, for the subspaces $IH^*_{\leq i}(X)$ of the perverse Leray filtration with respect to f, and for their graded pieces $IH_i(X)$. Given two finite filtrations, F and G on an object M of an abelian category, the Zassenhaus Lemma yields a canonical isomorphism $Gr^F_a Gr^G_b M \simeq Gr^G_b Gr^F_a M$. We apply this to the MHS $H^*(X')$ and obtain a canonical isomorphism $Gr^{F^g}_a Gr^{F^h}_b H^*(X') \simeq Gr^{F^h}_b Gr^{F^g}_a H^*(X')$ of MHS. Note that the decomposition theorem implies that each summand decomposes (as a vector space, a priori not as an MHS) according to the strata on Y of a common refinement of stratifications of the maps h and f, and that the Zassenhaus isomorphism is a direct sum map (of vector spaces, a priori not of MHS). We have the canonical

epimorphism of MHS

$$Gr_0^{F^g} H^*(X') \longrightarrow IH^*(X)$$

(this is how the MHS on the rhs has been constructed in the proof of Theorem 4.3.1 given in (3) above). This map induces the epimorphic map of MHS

$$Gr_i^{F^h} Gr_0^{F^g} H^*(X') \longrightarrow IH_i^*(X).$$

This map is a direct sum map with respect to the strata S on Y. It remains to show that each S-summand on the left-hand-side is a MHSS. Theorem 4.3.2, part (3) (which we have proved above for X' is nonsingular) implies that $Gr_i^{F^h} H^*(X')$ splits according to strata into MHSS. Each S-summand of this group maps onto the corresponding S-summand in $Gr_0^{F^g} Gr_i^{F^h} H^*(X')$ which is then an MHSS of this group. We conclude by the Zassenhaus Lemma, for the identification given by this lemma is compatible with the direct sum decomposition by strata and with the MHS.

(7) Theorem 4.3.2, part (4) is proved using the same argument employed in the case when X is nonsingular (see item (1) of this list), provided we replace $f_*\mathbb{Q}_X[n]$ with the self-dual f_*IC_X in Proposition 4.5.1.

(8) The proof of Theorem 4.3.3 in the case when X and Y are projective is the main result of [10]. The arguments provided in that paper are quite general and work verbatim in the quasi-projective case and also in the case of compact supports (one only has to replace the pure Hodge structures employed there, with the MHS introduced here).

(9) Intersection cohomology is not functorial in the "space" variable. The paper [2] constructs, for every proper map $f : X \to Y$, a noncanonical map $IH^*(Y) \to IH^*(X)$. If f is surjective, these morphisms stem from the decomposition theorem and are splitting injections. The same proof as [10], Theorem 3.4.1, again replacing pure with mixed, yields the proof of Theorem 4.3.4.

It is now easy to prove Theorem 4.3.5, i.e. to prove that the mixed Hodge structures we construct coincide with the ones arising from M. Saito's theory of mixed Hodge modules (MHM).

Proof of Theorem 4.3.5.

First proof. Let Y be a quasi-projective irreducible variety. By [22], the MHS on $H^*(Y)$ and $H_c^*(Y)$ stemming from the theory of MHM coincides with the one constructed by Deligne in [15]. It follows that if X is nonsingular, then the two kinds of MHS of the subspaces appearing in Theorem 4.3.2, part (3) coincide. We apply this fact to a resolution of the singularities $f : X \to Y$

of Y and we see that the two possible MHS on the intersection cohomology groups $IH^*(Y)$ and $IH^*_c(Y)$ of an irreducible quasi-projective variety coincide. Once the two MHS coincide on the intersection cohomology groups, they also coincide on the subspaces appearing in Theorem 4.3.2 (and now X can be singular).

Second proof. The splittings of Theorem 4.3.4 arrive from a construction in homological algebra (due to Deligne; see [10]) that works in the category $\mathcal{D}_Y$ as well as in the derived category $D^b(MHM_Y)$ of MHM on Y. This means that we can take the splitting ϕ in $\mathcal{D}_Y$ to be the trace of a splitting in $D^b(MHM_Y)$, and this implies the conclusion. $\square$

This ends the outline of the proofs. The next section is devoted to completing the proofs.

Let me try to give an idea of how the mixed Hodge-theoretic results are proved by looking at the following special simple case. Let $f : X \to Y$ be a proper birational map of surfaces, with X nonsingular, such that f is an isomorphism away from a curve $E \subseteq X$ contracted by f to a point on Y. The decomposition theorem implies that

$$H^2(X) = IH^2(Y) \oplus \langle [E] \rangle, \qquad H^2_c(X) = IH^2_c(Y) \oplus \langle [E] \rangle$$

where $[E]$ is the fundamental class of E.

Let me illustrate the technique used in this paper to verify that in both equations both summands are MHSS of the usual MHS.

Poincaré duality yields an isomorphism $\iota : H^2(X) \simeq H^2_c(X)^\vee$. The map ι is a direct sum map with respect to the direct sum decompositions above.

By invoking the decomposition theorem (see [8], proof of the purity theorem 2.2.1, especially (44)), the pull-back map in cohomology $r : H^2(X) \to H^2(E)$ is injective when restricted to the summand $\langle [E] \rangle$ and it is the zero map on $IH^2(Y)$. Since r is a map of MHS, its kernel $IH^2(Y)$ is an MHSS of the MHS $H^2(X)$.

We argue in the same way for cohomology with compact supports, except that the map r is now the map $r' : H^2_c(X) \to H^2_c(E)$ which is the dual of the proper push-forward map $H_2^{BM}(E) \to H_2^{BM}(X)$ in Borel-Moore homology. It follows that $IH^2_c(Y)$ is an MHSS of the MHS $H^2_c(X)$.

The Poincaré isomorphism is in fact an isomorphism, $\iota : H^2(X) \simeq H^2_c(X)^\vee(-2)$ of MHS.

The composition of maps of MHS $H^2(X) \to H^2_c(X)^\vee(-2) \to IH^2_c(Y)(-2)$ has kernel $\langle [E] \rangle$ which is then an MHSS of $H^2(X)$. Similarly, we show that $\langle [E] \rangle$ is an MHSS of $H^2_c(X)$.

4.5. Completion of the proofs of the results of §4.3.

Proposition 4.5.1. *Let $f : X \to Y$ be a proper map of irreducible quasi-projective varieties. Assume that X is nonsingular. For every i and j:*

(1) *the vector spaces $H^j_i(X)$ and $H^j_{c,i}(X)$ carry natural MHS which are subquotients of the canonical MHS $H^*(X, \mathbb{Q})$ and $H^*_c(X, \mathbb{Q})$.*
(2) *the Poincaré Pairing $H^j(X) \simeq H^{2m-j}_c(X)^\vee(-m)$ descends to an isomorphism of MHS*

$$H^j_i(X) \simeq H^{2m-j}_{c,-i}(X)^\vee(-m), \qquad H^j_{i,l,S}(X) \simeq H^{2m-j}_{c,-i,l,S}(X)^\vee(-m).$$

Proof. The spaces in question are the graded pieces of the perverse Leray filtration and part (1) follows from Theorem 4.1.1.

We turn to part (2). By the mixed Hodge theory of algebraic varieties (see [15]), the Poincaré pairings $H^j(X) \simeq H^{2m-j}_c(X)^\vee(-m)$ is an isomorphism of MHS. By [9], Lemma 2.9.1 (which proof, written for X proper and nonsingular, is valid when X is merely nonsingular), the Poincaré pairing above is compatible with the perverse Leray filtration, i.e. for $i \in \mathbb{Z}$ it induces maps $H^j_{\leq i}(X) \to H^{2m-j}_{c,\leq -i}(X)^\vee(-m)$, and it descends to each i-th graded group as a linear isomorphisms $P : H^j_i(X) \simeq H^{2m-j}_{c,-i}(X)^\vee(-m)$. This linear isomorphism is of MHS, for the subquotient MHS on the graded groups. This establishes the first statement of part (2), i.e. for $H^j_i(X)$.

We now turn to $H^j_{i,l,S}(X)$. By the same lemma quoted above, the linear isomorphism P coincides with the map in hypercohomology associated with the canonical isomorphism stemming from Verdier Duality (recall that $f_*\mathbb{Q}_X[m]$ is self-dual)

$${}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[m]) \simeq {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_{X[m]})^\vee.$$

Both sides split according to strata as in (4.2). We are left with showing that the map P is a direct sum map for this decomposition according to strata. This follows immediately from the fact that there are no nontrivial maps between intersection complexes supported on different subvarieties. □

The proof of the two lemmata below is a combination of the results of this paper and of the methods employed in [8] in the proof of the purity Theorem 2.2.1. The key new ingredient is Proposition 4.5.1. Another difference is that, since we need to argue using pairings, we need to simultaneously keep track of cohomology and of cohomology with compact supports, even if we are interested only in cohomology.

Even though we do not repeat the parts of the proof that are contained in [8], for the reader's convenience, in the course of the proof, we quote the relevant results from [8].

As in the proof of the purity Theorem 2.2.1 in [8], the proof is by induction $m := \dim X$. The first main step is carried out in the following lemma, where we deal with the cases $(i,j) \neq (0,m)$ as follows: (i) we take hyperplane sections of the domain X and deal with the cases when $i \neq 0$, and (ii) we take hyperplane sections of the target Y and deal with the cases $(i = 0, j \neq m)$. The remaining and crucial case when $(i = 0, j = m)$ is dealt-with in the second main step, Lemma 4.5.3.

Lemma 4.5.2. *If Theorem* 4.3.2*, part* (3) *holds for every projective map* $g : Z \to Z'$ *of quasi-projective varieties,* Z *nonsingular,* $\dim Z < \dim X$, *then it holds for every group* $H^j_i(X)$ *and* $H^j_{c,i}(X)$ *with* $(i,j) \neq (0,m)$.

Proof. See [8], Lemma 7.1.1. Choose a general hyperplane section $r : X^1 \to X$. In this proof we need the section to be smooth, so that we can apply induction, and transverse to the relevant stratifications, so that $r^! \simeq r^*[-2]$ when applied to complexes constructible with respect to those same stratifications. In what follows, we shall freely use these facts as well as that $f_* = f_!$, and $g_* = g_!$. We have the proper map $g : X^1 \to Y$ and the affine map $u : X \setminus X^1 \to Y$. There is the adjunction map

$$f_* \mathbb{Q}_X[m] \longrightarrow g_* \mathbb{Q}_{X^1}[m-1][1].$$

Taking hypercohomology and hypercohomology with compact supports, we obtain restriction maps

$$H^j_i(X) \longrightarrow H^j_{i+1}(X^1), \qquad H^j_{c,i}(X) \to H^j_{c,i+1}(X^1).$$

Similarly, by considering the adjunction map

$$g_* \mathbb{Q}_{X^1}[m-1][-1] \longrightarrow f_* \mathbb{Q}_X[m],$$

we obtain Gysin maps

$$H^{j-2}_{i-1}(X^1) \longrightarrow H^j_i(X), \qquad H^{j-2}_{c,i-1}(X^1) \longrightarrow H^j_c(X).$$

By the Weak-Lefschetz-type Proposition 4.7.6 in [8] (the key point is that u is affine, hence t-left exact), we have that the natural restriction-type map

$$^{\mathfrak{p}}\mathcal{H}^i((f_* \mathbb{Q}_X[m])) \longrightarrow \; ^{\mathfrak{p}}\mathcal{H}^{i+1}((g_* \mathbb{Q}_{X^1}[m-1]))$$

is a splitting monomorphism for $i < 0$, and that the natural Gysin-type map

$$^{\mathfrak{p}}\mathcal{H}^{i-1}((g_* \mathbb{Q}_{X^1}[m-1])) \longrightarrow \; ^{\mathfrak{p}}\mathcal{H}^i((f_* \mathbb{Q}_X[m]))$$

is a splitting epimorphism for $i > 0$.

Since the restriction and Gysin maps above are direct sum maps with respect to the direct sum decomposition by strata, the statement of the lemma follows for every (i,j) with $i \neq 0$ by virtue of the hypotheses applied to $Z = X^1 \to Y = Z'$ (see [8], p. 744, bottom).

Let $i = 0$ and $j \neq m$. The argument is formally similar to the one just given. However, instead of using hyperplane sections of X and Weak-Lefschetz-type results for the perverse cohomology complexes ${}^{\mathfrak{p}}\mathcal{H}^i((f_*\mathbb{Q}_X[m]))$, $i \neq 0$, we work with hyperplane sections on Y and Weak-Lefschetz-type results on the cohomology groups

$$H^j(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[m])), \qquad H^j_c(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[m])), \qquad \forall j \neq 0.$$

We need to show that the theorem holds for the following four groups.

$$(1)\ H_0^{j<m}(X), \quad (2)\ H_{c,0}^{j>m}(X), \quad (3)\ H_0^{j>m}(X), \quad (4)\ H_{c,0}^{j<m}(X).$$

The cases (1) and (2) are dual to each other and so are (3) and (4). It follows that it is enough to establish the result in cases (1) and (3).

We choose an affine embedding Y into projective space and a general hyperplane section Y_1 of Y with respect to this embedding. Note that, in this case, $Y \setminus Y_1$ is affine. We have the associated map $h : X_1 := f^{-1}(Y_1) \to Y_1$ and the decomposition theorem for h takes the form of the decomposition (4.1) restricted to Y_1 and shifted by $[-1]$ and we have

$${}^{\mathfrak{p}}\mathcal{H}^i(h_*\mathbb{Q}_{X_1}[m-1]) = {}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[m])_{|Y_1}[-1].$$

Note that the skyscraper summands disappear after restriction (though this plays no role in the sequel of this proof).

Case (1). The natural restriction maps $H^j(X) \to H^j(X_1)$ are of MHS. It follows that the maps $H_0^j(X) \to H_0^j(X_1)$ induced on the subquotients are of MHS with respect to the MHS stemming from Proposition 4.5.1. Moreover, these induced maps are direct sum maps with respect to the direct sum decompositions by strata (4.4). By the inductive hypothesis, the theorem holds for $h : X_1 \to Y_1$. It is thus enough to show that the natural restriction maps above are injective for $j < m$. Since the Lefschetz hyperplane Theorem 2.0.3 applied to the perverse sheaf ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[m])$ on Y implies injectivity in the desired range, the result follows in case (1).

Case (3). The restriction map (see Remark 2.0.4)

$$H^j(X_1, i^!\mathbb{Q}) = H^{j-2}(X_1, \mathbb{Q}) \longrightarrow H^j(X, \mathbb{Q})$$

is the natural Gysin map and is of MHS. Passing to graded groups, the induced Gysin-type map $H_0^{j-2}(X_1) \to H_0^j(X)$ is a map of MHS and it is a direct sum map with respect to the decomposition by strata. As in case (1), it is enough to show that this Gysin-type map is surjective for $j > m$. The Gysin-type map in question appears in the long exact sequence of cohomology of the triangle

$$I_! I^! \, {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[m]) \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[m]) \longrightarrow J_* J^* \, {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[m]) \xrightarrow{+}$$

on Y, where $I : Y_1 \to Y \leftarrow Y - Y_1 : J$ are the natural inclusions. To establish the wanted surjectivity, it is enough to observe that

$$H^r(Y, J_* J^* \, {}^{\mathfrak{p}}\mathcal{H}^0(f_* \mathbb{Q}_X[m])) = H^r(Y - Y_1, J^* \, {}^{\mathfrak{p}}\mathcal{H}^0(f_* \mathbb{Q}_X[m])) = 0$$

by the theorem on the cohomological dimension of affine varieties for perverse sheaves applied to the perverse $J^* \, {}^{\mathfrak{p}}\mathcal{H}^0(f_* \mathbb{Q}_X[m])$. The result follows also in case (3). □

The following lemma takes care of the remaining cases $H^m_0(X)$ and $H^m_{c,0}(X)$ and completes the proof of Theorem 4.3.2, part (3), in the case when X is nonsingular.

Lemma 4.5.3. *Theorem* 4.3.2*, part* (3) *holds when X is nonsingular.*

Proof. The proof is by induction on $\dim X$. The cases $\dim X = 0, 1$ are trivial.

We assume that the theorem holds for every map $g : Z \to Z'$ as in Lemma 4.5.2. By the same lemma, we are left with the cases of $H^m_0(X)$ and $H^m_{c,0}(X)$.

We choose a nondense stratum S in $f(X)$ and we proceed exactly as in the proof of Theorem 2.2.1 in [8] and we prove that

$$\bigoplus_{l, S' \neq S} H^m_{0,l,S'}(X) \subseteq H^m_0(X) \tag{4.6}$$

is an MHSS. The only minor difference is that when we take the closure Z'_S of $f^{-1}(S)$ in X and take a resolution of the singularities $\rho : Z_S \to Z'_S$, the resulting quasi-projective variety Z_S is not projective, however, the properness of Z_S plays no role in [8]; what is essential is the fact that $Z'_S \to \overline{S}$ is proper. Let us point out that we can use the inductive hypothesis in view of the fact that $\dim Z_S < \dim X$, and this explains why we started with a nondense stratum.

By taking intersections, we see that any direct sum of terms which includes the dense stratum $\Sigma \subseteq f(X)$ gives an MHSS of $H^m_0(X)$.

The same line of reasoning works for $H^m_{c,0}(X)$. The only difference is that the maps we use are not pull-back maps in cohomology, but rather the maps in cohomology with compact supports which are the duals of the proper push-forward maps in Borel-Moore homology.

We are left with the case of the summands associated with the dense stratum Σ in $f(X)$. Consider the composition of maps of MHS (dualizing turns an MHSS into a quotient MHS):

$$H^m_0(X) \simeq H^m_{c,0}(X)^\vee(-m) \longrightarrow \bigoplus_{l, S' \neq \Sigma} H^m_{c,0,l,S'}(X)^\vee(-m).$$

The kernel, $H^m_{0,\dim\Sigma,\Sigma}(X)$, is an MHSS of $H^m_0(X)$. It follows that all direct summands $H^m_{0,l,S}(X)$, of $H^m_0(X)$ are MHSS. By dualizing, the same is true for $H^m_{c,0}(X)$ and its summands. □

The following is the intersection cohomology analogue of Theorem 4.1.1. It is needed to endow the perverse cohomology groups $IH^*_i(X)$ and $IH^*_{c,i}(X)$ with MHS. For convenience only, in the proof below, we use a different normalization for the intersection complex, i.e. if X is an irreducible variety, then set $\mathcal{I}C_X := IC_X[-\dim X]$ so that $IH^j(X) = H^j(X, \mathcal{I}C_X)$. If X is irreducible and nonsingular, then $\mathcal{I}C_X = \mathbb{Q}_X$.

Proposition 4.5.4 (Perverse Leray and MHS: intersection cohomology)**.** *Let $f : X \to Y$ be a map of irreducible quasi-projective varieties. There is an integer d, a variety $\mathcal{X}$, and a flag $\mathcal{X}_*$ on it such that there are identities of filtered groups*

$$(IH^*(X), L^f_{\mathfrak{p}_\tau}) = (IH^*(\mathcal{X}), F_{\mathcal{X}_*}(-d)), \quad (IH^*_c(X), L^f_{\mathfrak{p}_\tau}) = (IH^*_c(\mathcal{X}), G_{\mathcal{X}_*}(d)).$$

In particular, the subspaces of the perverse Leray filtrations on $IH^*(X)$ and $IH^*_c(X)$ are MHSS (for the MHS of Theorem 4.3.1*), and the graded groups $IH_i(X)$ and $IH_{c,i}(X)$ inherit the subquotient MHS.*

Proof. We employ the set-up of §3.2.3, especially (3.21). Note that $\pi^*\mathcal{I}C_X = \mathcal{I}C_{\mathcal{X}}$. The identities of filtered groups stem from Theorem 3.3.5 applied to $C = \mathcal{I}C_X$.

Claim. Let $i : \mathcal{H} \to \mathcal{Y}$ be a general codimension c linear section (for any embedding in projective space). Let $i : \mathcal{X}_H \to \mathcal{H}$ be the pre-image of $\mathcal{H}$. The restriction map $IH^*(\mathcal{X}) \to IH^*(\mathcal{X}_{\mathcal{H}})$ and the co-restriction map $IH^*_c(\mathcal{X}_{\mathcal{H}}) \to IH^*_c(\mathcal{X})$ are maps of MHS for the MHS of Theorem 4.3.1.

Proof of the Claim. Let $h : \mathcal{X}' \to \mathcal{X}$ be a resolution of the singularities of $\mathcal{X}$. We have the Cartesian diagram:

$$\begin{tikzcd}
\mathcal{X}'_{\mathcal{H}} \arrow[r, "h"] \arrow[d, "i"] \arrow[rr, bend left, "f':=f\circ h"] & \mathcal{X}_{\mathcal{H}} \arrow[r, "f"] \arrow[d, "i"] & \mathcal{H} \arrow[d, "i"] \\
\mathcal{X}' \arrow[r, "h"] \arrow[rr, bend right, "f':=f\circ h"'] & \mathcal{X} \arrow[r, "f"] & \mathcal{Y}.
\end{tikzcd}$$

For $\mathcal{H}$ sufficiently general, $\mathcal{I}C_{\mathcal{X}_{\mathcal{H}}} = i^*\mathcal{I}C_{\mathcal{X}}$ and $f' : \mathcal{X}'_{\mathcal{H}} \to \mathcal{X}_{\mathcal{H}}$ is also a resolution.

By the decomposition theorem, the intersection complex $\mathcal{I}C_{\mathcal{X}}$ is a direct summand of $h_*\mathbb{Q}_{\mathcal{X}'}$ (see 4.1). Similarly, for $\mathcal{I}C_{\mathcal{X}_{\mathcal{H}}}$ and $h_*\mathbb{Q}_{\mathcal{X}'_{\mathcal{H}}}$. The two direct sum decompositions are related by the restriction map $h_*\mathbb{Q}_{\mathcal{X}'} \to i_*h_*\mathbb{Q}_{\mathcal{X}'_{\mathcal{H}}}$.

This map is a direct sum map and maps $\mathcal{IC}_{\mathcal{X}}$ to $\mathcal{IC}_{\mathcal{X}_{\mathcal{H}}}$. This map yields the restriction map in intersection cohomology $IH^*(\mathcal{X}) \to IH^*(\mathcal{X}_{\mathcal{H}})$. Similarly, for the co-restriciton map.

We complete the proof of the CLAIM for the restriction map. The case of the co-restriction map is completely analogous. The restriction map $H^*(\mathcal{X}') \to H^*(\mathcal{X}'_{\mathcal{H}})$ is of MHS. The spaces $H_{\leq 0}(\mathcal{X}) \subseteq H^*(\mathcal{X})$ and $H_{\leq 0}(\mathcal{X}'_{\mathcal{H}}) \subseteq H_{\leq 0}(\mathcal{X}'_{\mathcal{H}})$ are MHSS, mapped into each other via the restriction map. It follows that the restriction map descends to a map $H_0(\mathcal{X}') \to H_0(\mathcal{X}'_{\mathcal{H}})$ of MHS. This map is a direct sum map with respect to the strata and, by Theorem 4.3.2, part (3), each component is a map of MHS. Since the restriction map in intersection cohomology is one of these summands, the CLAIM follows.

We conclude the proof by observing that the flag $\mathcal{X}_*$ is a pull-back flag of a general flag $\mathcal{Y}_*$ on $\mathcal{Y}$ and, by applying the CLAIM to the elements of the flag, we see that the kernels (images, resp.) of the restriction (co-restriction, resp.) maps, i.e. the subspaces of the perverse filtrations, are MHSS. □

4.6. The results of §4.3 hold for not necessarily irreducible varieties. In this section we point out that, while we have stated most of the Hodge-theoretic applications in §4.3 in the context of quasi-projective *irreducible* varieties, these results in fact hold for quasi-projective varieties.

The key point is to give the correct definition of intersection complex for not necessarily irreducible, nor pure dimensional varieties. Once that definition is in place, the rest follows quite easily.

Given an irreducible variety Y of dimension n, one defines the intersection complex IC_Y as follows: let $j : U \subseteq Y$ be a Zariski-dense, open and nonsingular subset of Y; then $IC_Y = j_{!*}\mathbb{Q}_U[n]$ is the intermediate extension of the perverse sheaf $\mathbb{Q}_U[n]$.

Let Y be any variety and $Y_{reg} = \coprod_{l \geq 0} U_l$ be the decomposition of the regular part in pure l-dimensional components. Let $\mathfrak{Q} := \bigoplus_{l \geq 0} \mathbb{Q}_{U_l}[l]$ and $j : Y_{reg} \to Y$ be the open immersion. Define the intersection complex of Y as follows

$$IC_Y = j_{!*}\mathfrak{Q}.$$

The following are easily verified:

(1) $IC_Y = \bigoplus_{l \geq 0} IC_{\overline{U}_l}$;
(2) let $\nu : Y' \to Y$ be the normalization; recall that Y' is a disjoint union of irreducible normal varieties and that ν is finite, so that $\nu_* = R^0\nu_*$; we have $\nu_* IC_{Y'} = IC_Y$.

It follows that with this definition of intersection complex the decomposition theorem holds for a proper map of varieties. In fact, one normalizes the domain and works on each component separately.

It is easy to generalize all the mixed-Hodge-theoretic applications in §4.3 of this paper to arbitrary quasi-projective varieties, provided we use the definition of intersection complex given above.

We leave to the reader the task of verifying that all the proofs go through verbatim, with the possible exception of the one of Theorem 4.3.1, part (3).

In this case, we need to verify that the map $a : H^*(Y) \to IH^*(Y)$ is of MHS. This is done by reduction to the irreducible case as follows. Let $\{Y_t\}$ be the set of irreducible components of Y. We have that $IH^*(Y) = \bigoplus_t IH^*(Y_t)$. The map a is of MHS iff the induced maps $a_t : H^*(Y) \to IH^*(Y_t)$ are all of MHS. The map a_t factors as follows $H^*(Y) \to H^*(Y_t) \to IH^*(Y_t)$. The first map is of MHS by the theory of MHS for cohomology. The second one is of MHS by Theorem 4.3.1, part (3).

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