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The Chow Groups and the Motive of the Hilbert Scheme of Points on a Surface

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We compute the Chow motive and the Chow groups with rational coefficients of the Hilbert scheme of points on a smooth algebraic surface. © 2002 Elsevier Science (USA)

1. INTRODUCTION

In this paper we compute the Chow motive and the Chow groups with rational coefficients of the Hilbert scheme $X^{[n]}$ of n points on a nonsingular algebraic surface X. The result holds over an arbitrary field.

The space $X^{[n]}$ is a remarkably rich object which finds itself, for reasons that are perhaps still mysterious, at the crossroads of geometry,

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representation theory, and mathematical physics. In the context of testing the S-duality conjecture, using Göttsche's calculation of the Betti numbers of $X^{[n]}$, Vafa and Witten suggested, for reasons stemming from orbifold cohomology, that the singular cohomology groups of these Hilbert schemes should be naturally linked to infinite dimensional graded Lie algebras. This fact was firmly established by Nakajima and by Grojnowski, independently. Some of the contributors to this circle of ideas are Fogarty [7], Briançon [1], Iarrobino [14], Ellingsrud and Stromme [5], Göttsche [9], Göttsche and Soergel [11], Cheah [3], Vafa and Witten [18], Nakajima [16], and Grojnowski [12]. The reader is referred to the beautiful lectures [17] and to our paper [4] for background, further results, and references to the literature.

In our paper [4] we proved, directly, a precise form of the decomposition theorem for the so-called Hilbert–Chow map. We detected, via the action of the Lie algebra, certain subvarieties of $X^{[n]}$ carrying the topological information necessary to understand the additive structure of singular cohomology.

The Grojnowski and Nakajima approach to the singular cohomology groups of Hilbert schemes in terms of correspondences and its relation to infinite dimensional Lie algebras has played an important role in the understanding of these configuration spaces.

The present paper introduces two sets of new correspondences associated with Hilbert schemes and uses them to compute Chow groups and Chow motives. The Grojnowski–Nakajima picture and approach do not seem to apply to the context of this paper. See Remark 5.4.5. In this paper we show how these correspondences identify the Chow groups with rational coefficients of $X^{[n]}$ with the Chow groups with rational coefficients of a certain collection of products of symmetric products of the surface X. Using this result, we determine the Chow motive with rational coefficients of $X^{[n]}$.

Voevodsky's theories of motivic cohomology incorporate the formalisms of Tate twists and shifts of complexes. Once the Chow motive of $X^{[n]}$ is computed, it is easy to compute the motive in these more general theories. Our result, read in these theories, harmonizes the shifts present in the decomposition theorem mentioned above and the Tate twists in the mixed Hodge structure of $X^{[n]}$. There is also a generating function for the motives in question.

The theory of Grothendieck motives has been introduced by Grothendieck as the archetypical cohomology theory. See [15] and [8, 16.1, especially 16.1.12]. Very roughly and in brief, every good cohomology theory should factor through this one. The theory of motives requires to consider pairs (X, p), where X is a nonsingular projective variety and $p \in A_{\dim X}(X \times X)$ is a cycle with rational coefficients modulo some equivalence relation, such that, in the formalism of correspondences, $p \circ p = p$.

The usual yoga associates with (X, p) a direct summand of $A_*(X)$, namely $p_*(A_*(X))$. Note that for (X, Δ_X), $\Delta_{X*}(A_*(X)) = A_*(X)$. Grothendieck's theory corresponds to the coarsest possible equivalence relation: numerical equivalence. The standard motivic conjectures, are, roughly, the necessary properties that cycles should enjoy, for this theory to be the universal cohomology theory in the sense hinted at above. One may also take the a priori finer algebraic equivalence or the even more refined linear equivalence. In this latter case one gets Chow motives. A result proved for Chow motives implies the analogous fact for Grothendieck motives and, since this latter cohomology theory specializes to singular cohomology, it also implies the analogous result for singular cohomology. In general the converse is, by far, not true.

One way to prove structure results for $A_*(X)$, for example, is to determine a natural and nontrivial decomposition of the diagonal $\Delta_X = \sum_k p_k$ (with respect to one of the equivalence relations) subject to the orthogonality conditions $p_i \circ p_j = \delta_{ij} p_i$. In that case, $A_*(X) = \sum_k p_{k*}(A_*(X))$. This would imply structure results for Chow groups, singular cohomology, mixed Hodge structure, K-theory, etc.

In this paper we find such a natural decomposition of the diagonal of $X^{[n]}$ with respect to the most refined equivalence relation: linear equivalence; see Proposition 6.1.5. To accomplish this, we first compute the Chow groups; see Theorem 5.4.1. The structure of the Chow motive follows; see Theorem 6.2.1.

These structure theorems for Chow motives and for Chow groups are new. They imply the known results on the additive structure of the singular cohomology of $X^{[n]}$, but are not implied by them. They are valid for a not necessarily complete surface over an arbitrary field.

The main tool used in our calculation is the Gysin formalism of Fulton–MacPherson. The fact, due to Ellingsrud and Stromme, that punctual Hilbert schemes admit affine cellular decompositions is essential to our approach.

The outline of the paper is as follows. Section 2 is devoted to fixing the notation and to introducing the varieties naturally associated with $X^{[n]}$ and its natural stratification given by partitions. Section 3 reviews the basic results from intersection theory that we need. Section 4 reviews well-known facts concerning the Gysin formalism of correspondences in a "non-complete" situation. Section 4.1 discusses correspondences in this "refined" formalism. Section 4.2 discusses the composition of correspondences in the context of the refined formalism. Section 4.3 overviews the situation for quotient varieties. In Section 4.4 we define the natural map $\widehat{\Gamma}_*$ via the correspondences $\widehat{\Gamma}$. The main result of the paper is Theorem 5.4.1, which states that $\widehat{\Gamma}_*$ is an isomorphism. The injectivity statement is Corollary 5.1.5. The surjectivity statement, Corollary 5.3.2, is the heart of the present paper. Corollary 5.4.2 and Remark 5.4.3 are standard consequences for the Grothendieck groups also in the equivariant context. Section 6 is devoted to the identification, Theorem 6.2.1, of the motive of the Hilbert scheme $X^{[n]}$ with a collection of motives of products of symmetric products of the surface X. Theorem 6.2.5 is the translation of Theorem 6.2.1 into Voevodsky's categories. We give a "generating function" for this motivic structure at the end of Section 6.1.2.

Finally, Remark 5.4.3 and Remark 6.2.4 discuss the relation of this paper with recent work of Haiman, Göttsche, and Bridgeland, King, and Reid.

2. SOME SPECIAL VARIETIES AND FIBRATIONS

In this section we review a few definitions and introduce the correspondences which will play a major role in the paper. Proofs or references for all the results stated here can be found in our previous paper [4]. Let X be an irreducible quasi-projective nonsingular surface defined over an algebraically closed field, let $X^{(n)}$ be its nth symmetric product, let $X^{[n]}$ be the Hilbert scheme of 0-dimensional subschemes of X of length n, and let π: $X^{[n]} \to X^{(n)}$ be the Hilbert–Chow morphism. It is well known that $X^{[n]}$ is nonsingular. $\mathfrak{P}(n)$ denotes the set of partitions of n and $p(n)$ denotes its cardinality. If $\nu \in \mathfrak{P}(n)$, we denote by $l(\nu)$ its length, and define $X^{(n)}_\nu$ to be the locally closed subset of points in $X^{(n)}$ of the type $\nu_1 x_1 + \cdots + \nu_{l(\nu)} x_{l(\nu)}$, with $x_h \in X$ and $x_i \neq x_j$ for every $i \neq j$. Similarly, $X^{[n]}_\nu := (\pi^{-1}(X^{(n)}_\nu))_{red}$. Let $\overline{X}^{(n)}_\nu$ denote the closure of the stratum. It can be proved that $\overline{X}^{[n]}_\nu = (\pi^{-1}(\overline{X}^{(n)}_\nu))_{red}$. If $\nu = 1^{a_1} \cdots n^{a_n}$, then the finite group $\Sigma_\nu := \Sigma_{a_1} \times \cdots \times \Sigma_{a_n}$ acts naturally on $X^{l(\nu)}$. The quotient $X^{(\nu)}$ is isomorphic to $X^{(a_1)} \times \cdots \times X^{(a_n)}$. For the sake of uniformity of notation, we shall denote $X^{l(\nu)}$ by X^ν. There is a natural Σ_ν-invariant map ν: $X^\nu \to X^{(n)}$ whose image is $\overline{X}^{(n)}_\nu$. This map descends to a map which we denote by the same symbol ν: $X^{(\nu)} \to X^{(n)}$. This map is the normalization map of $\overline{X}^{(n)}_\nu$. We define correspondences Γ^ν and $\widehat{\Gamma}^\nu$ as follows:

$$\begin{aligned}\Gamma^\nu &= \left\{(x_1, \ldots, x_{l(\nu)}, \mathcal{J}) \in X^\nu \times X^{[n]} \colon \pi(\mathcal{J}) = \nu_1 x_1 + \cdots + \nu_{l(\nu)} x_{l(\nu)}\right\} \\ &\simeq \left(X^\nu \times_{X^{(n)}} X^{[n]}\right)_{red}.\end{aligned}$$

Denote by X^ν_{reg} (resp. $X^{(\nu)}_{reg}$) the set of points of X^ν (resp. $X^{(\nu)}$) strictly of type ν.

Remark 2.0.1. The restriction Γ^ν_{reg} of Γ^ν to X^ν_{reg} is a Zariski locally trivial fibration with irreducible fibers and it is open and dense in Γ^ν (see [4, Lemma 3.6.1] and Remark 5.2.5). It follows that Γ^ν is irreducible of dimension $n + l(\nu)$.

The projection p_1 will be denoted by p_ν in order to emphasize the dependence on ν. The correspondence Γ^ν is invariant under the action of Σ_ν on the first factor of the product. We can therefore define $\widehat{\Gamma}^\nu := \Gamma^\nu/\Sigma_\nu$. We summarize these constructions with the diagram

$$\begin{tikzcd} \Gamma^\nu \arrow[r, "q'"] \arrow[d, "p_\nu"'] & \widehat{\Gamma}^\nu \arrow[r, "P"] \arrow[d, "\hat{p}_\nu"'] & X^{[n]} \arrow[d, "\pi"'] \\ X^\nu \arrow[r, "q"] & X^{(\nu)} \arrow[r, "\nu"] & X^{(n)} \end{tikzcd},$$

where q and q' are the quotient maps by the action of Σ_ν. The composition $P \circ q'$ will still be denoted by P.

Remark 2.0.2. The restriction of P to $\widehat{\Gamma}^\nu_{reg} = \Gamma^\nu_{reg}/\Sigma_\nu$ identifies $\widehat{\Gamma}^\nu_{reg}$ with the locally closed stratum $X^{[n]}_\nu$.

We introduce $\mathscr{X} = \coprod_{\nu \in \mathfrak{P}(n)} X^{(\nu)}$ and $\widehat{\mathscr{X}} = \coprod_{\nu \in \mathfrak{P}(n)} X^{(\nu)}$, $\Gamma = \coprod_{\nu \in \mathfrak{P}(n)} \Gamma^{(\nu)}$, $\widehat{\Gamma} = \coprod_{\nu \in \mathfrak{P}(n)} \widehat{\Gamma}^\nu$, and define $p: \Gamma \to \mathscr{X}$ (resp. $\hat{p}: \widehat{\Gamma} \to \widehat{\mathscr{X}}$) to be the map induced by all the maps p_ν (resp. $\hat{p}_\nu$). The set of partitions has a natural partial order, which reflects the incidence relations of the strata:

DEFINITION 2.0.3. Let $\nu, \mu \in \mathfrak{P}(n)$. We say that $\nu \succeq \mu$ if there exists a decomposition $\{I_1, \ldots, I_{l(\mu)}\}$ of the set $\{1, \ldots l(\nu)\}$ such that $\mu_1 = \sum_{i \in I_1} \nu_i, \ldots, \mu_{l(\mu)} = \sum_{i \in I_{l(\mu)}} \nu_i$.

It is easily seen that $\mu \succeq \nu$ if and only if $X^{(n)}_\nu \subseteq \overline{X}^{(n)}_\mu$. Fix a total order $\geq$ on $\mathfrak{P}(n)$ which is compatible with $\succeq$. For any $\mu \in \mathfrak{P}(n)$, we have the open subsets $X^{(n)}_{\geq\mu} := \coprod_{\nu \geq \mu} X^{(n)}_\nu \subseteq X^{(n)}$. Similarly for $X^{(n)}_{>\mu}$. Correspondingly, we have open sets $\mathscr{X}_{\geq\mu}$, $\mathscr{X}_{>\mu}$, $\Gamma_{\geq\mu}$, $\Gamma_{>\mu}$, $X^{[n]}_{\geq\mu}$, and $X^{[n]}_{>\mu}$ obtained by base change followed by reduction. We have the corresponding quotients $\widehat{\mathscr{X}}_{\geq\mu}$, $\widehat{\mathscr{X}}_{>\mu}$, $\widehat{\Gamma}_{\geq\mu}$, $\widehat{\Gamma}_{>\mu}$. Similarly, with the symbol $\geq$ replaced by $\succeq$.

The imbedding $X^{(n)}_\mu \to X^{(n)}_{\geq\mu}$ is closed. We have corresponding closed imbeddings $\mathscr{X}_\mu \to \mathscr{X}_{\geq\mu}$ and $\Gamma_\mu \to \Gamma_{\geq\mu}$ obtained by base change followed by reduction. Note that $\Gamma_\mu \neq \Gamma^\mu$. We thus have a diagram:

$$\begin{tikzcd}
\Gamma_\mu \arrow[rr, "P"] \arrow[dr] \arrow[dd, "p_{\geq\mu}"'] & & X_\mu^{[n]} \arrow[dr] \arrow[dd, "\pi", near start] & \\
& \Gamma_{\geq\mu} \arrow[rr, crossing over, "P", near start] & & X_{\geq\mu}^{[n]} \arrow[dd, "\pi"] \\
\mathscr{X}_\mu \arrow[rr] \arrow[dr] & & X_\mu^{(n)} \arrow[dr] & \\
& \mathscr{X}_{\geq\mu} \arrow[rr] \arrow[from=uu, crossing over, "p_{\geq\mu}", near start] & & X_{\geq\mu}^{(n)}
\end{tikzcd}.$$

3. REVIEW OF INTERSECTION THEORY

Our unique reference for this section is [8]. In what follows, the Chow groups $A_*(X)$ of an algebraic scheme X over a field are always taken with rational coefficients, even though many results hold with integer coefficients. We recall that, given a regular imbedding $i\colon X \to Y$ of codimension d with normal bundle $N_X Y$ and a morphism $f\colon Y' \to Y$ from a pure l-dimensional variety Y', there are refined Gysin homomorphisms $i^!\colon A_*(Y') \to A_{*-d}X'$, where $X' = X \times_Y Y'$. The construction of $i^!([Y'])$ goes as follows: the map $X' \to Y'$ is a closed imbedding and the normal cone $C_{X'}Y'$ is a pure l-dimensional subscheme of the pullback of $N_X Y$ to X'; its cycle class is therefore equivalent to the flat pullback of a unique cycle class $i^!([Y']) \in A_{l-d}(X')$. To define $i^!(\alpha)$ for $\alpha \in A_*(Y')$, one replaces Y' with the support of α and maps the resulting class to $A_*(Y')$.

Let $f\colon X \to Y$ be a morphism from a scheme X to a nonsingular variety Y. The graph morphism $\gamma_f\colon X \to X \times Y$ is a regular imbedding. Given maps $X' \to X$ and $Y' \to Y$, there is a refined Gysin morphism $\gamma_f^!\colon A_k(Y') \otimes A_l(X') \to A_{k+l-\dim Y}(X' \times_Y Y')$. If $X' = X$, $Y' = Y$, and both maps are the identity map, then we will denote $\gamma_f^!(\alpha \otimes \beta)$, the image of $\alpha \otimes \beta$ via the morphism $A_k(Y) \otimes A_l(X) \to A_{k+l-\dim Y}(X)$, with the more suggestive piece of notation $f^*(\alpha) \cap \beta$. If $X' = X$, then we will use the notation $f^!(\alpha) \cap \beta$ for the refined intersection.

Remark 3.0.1. Let $\alpha \in Z_k(Y')$, $\beta \in Z_l(X')$ be irreducible cycles. If the irreducible components ξ_i of $|\alpha| \times_Y |\beta|$ have the expected dimension $k + l - \dim Y$, then $\gamma_f^!(\alpha \otimes \beta)$ is a linear combination with strictly positive coefficients of the cycles ξ_i; see [8, Sect. 7.1]. In particular, if Y' is a closed subvariety of Y and $f^{-1}(Y')$ is irreducible of dimension $\dim Y' + \dim X - \dim Y$, then $f^!([Y']) \cap [X]$ is a positive multiple of $[f^{-1}(Y')]$.

The following well-known lemma will be used in the sequel of the paper. We omit the simple proof, which follows easily from the case of a trivial fibration (see [8, Ex. 1.10.2]) and by noetherian induction using the basic properties of the refined Gysin formalism.

LEMMA 3.0.2. *Let X be a nonsingular irreducible variety and let $p\colon \Gamma \to X$ be a Zariski locally trivial fibration with fiber F admitting a cellular decomposition. Suppose $\{\alpha_i\}_{i\in I}$ is a set of classes in $A_*(\Gamma)$ whose restrictions generate $A_*(p^{-1}(x))$ for every $x \in X$. Then $\{\alpha_i\}_{i\in I}$ is a set of generators of the $A_*(X)$-module $A_*(\Gamma)$. In other words, the map $\Phi\colon A_*(X)^{\oplus I} \to A_*(\Gamma)$, defined by $\Phi(\{\beta_i\}) = \sum_i p^*(\beta_i) \cap \alpha_i$, is surjective.*

4. CORRESPONDENCES

4.1. *Correspondences via Refined Gysin Maps*

The standard reference is [8, Remark 16.1, Sects 6 and 8]. Consider a diagram

$$\begin{tikzcd} \Gamma \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_2 \\ X_1 & \end{tikzcd}$$

with X_1 nonsingular and p_2 proper. In Section 3 we define a map

$$A_k(X_1) \otimes A_l(\Gamma) \xrightarrow{p_1^*(-)\cap(-)} A_{k+l-\dim X_1}(\Gamma) \xrightarrow{p_{2*}} A_{k+l-\dim X_1}(X_2).$$

The case we will mostly use is

$$p_{2*}(p_1^*(-) \cap [\Gamma])\colon A_k(X_1) \longrightarrow A_{k+\dim\Gamma-\dim X_1}(X_2),$$

where $[\Gamma]$ is some fixed cycle.

Most important for us will be the case of relative correspondences of the type

$$\begin{tikzcd} \Gamma \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & X_2 \arrow[d] \\ X_1 \arrow[r, "f"] & Y \end{tikzcd},$$

where X_1 is a nonsingular variety, f is a proper morphism, and $\Gamma = X_1 \times_Y X_2$. If $V \to Y$ is a morphism, then we have a similar diagram by pulling back to V. Set, for brevity, $X_{1,V} = X_1 \times_Y V$ and similarly $X_{2,V} = X_2 \times_Y V$ and $\Gamma_V = \Gamma \times_Y V$. Note that, by the transitivity property of the fiber product, $\Gamma_V = X_{1,V} \times_V X_{2,V}$. Fix a cycle $[\Gamma]$. The refined intersection product gives a map $p_1^!(-) \cap [\Gamma]$: $A_*(X_{1,V}) \to A_*(\Gamma_V)$.

Remark 4.1.1. It is important that we do not assume that X_1 is complete, i.e., that we do not assume that p_2: $X_1 \times X_2 \to X_2$ is proper and that we do not factorize through $A_*(X_1 \times X_2)$. Otherwise, we would lose too much information; e.g., the diagonal in $\mathbb{A}^1 \times \mathbb{A}^1$ is zero in $A_1(\mathbb{A}^1 \times \mathbb{A}^1)$, but it induces the identity map via the formalism of refined intersections.

Let i: $Z \to Y$ be a closed imbedding and let j: $U := Y \backslash Z \to Y$ be the resulting open imbedding.

LEMMA 4.1.2. *Let* $[\Gamma]$ *be any cycle in* $A_*(\Gamma)$. *The diagram*

$$\begin{tikzcd}
A_*(X_{1,Z}) \arrow[r, "p_1^!(-)\cap[\Gamma]"] \arrow[d, "i_*"'] & A_*(\Gamma_Z) \arrow[r, "p_{2*}"] \arrow[d] & A_*(X_{2,Z}) \arrow[d] \\
A_*(X_1) \arrow[r, "p_1^*(-)\cap[\Gamma]"] \arrow[d, "j^*"'] & A_*(\Gamma) \arrow[r, "p_{2*}"] \arrow[d] & A_*(X_2) \arrow[d] \\
A_*(X_{1,U}) \arrow[r, "p_1^*(-)\cap[\Gamma_U]"] \arrow[d] & A_*(\Gamma_U) \arrow[r, "p_{2*}"] \arrow[d] & A_*(X_{2,U}) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

is commutative and the columns are exact.

Proof. The exactness of the columns stems from [8, Proposition 1.8]. The commutativity of the first column of diagrams stems from [8, Proposition 8.1.1(c) and Theorem 6.2]. The commutativity of the second column is obvious. ∎

Remark 4.1.3. Note that, even if Γ is reduced, Γ_Z may fail to be so. However, this causes no trouble in view of the obvious canonical isomorphism $A_*(-_{red}) \to A_*(-)$. In the sequel we shall often take fiber products.

We shall always assume that we are taking the reduced structure. When we write that a diagram is "cartesian modulo nilpotents," we mean that the fiber product is to be taken with the reduced structure. We shall always take care to define the cycle $[\Gamma]$.

4.2. *Composition of Correspondences*

This formalism of correspondences via refined Gysin maps extends easily to the case of composition of correspondences. The reference is [8, Remark 16.1, Proposition 16.1.2, Sections 6 and 8].

Let X_i, $i = 1, 2, 3$, be nonsingular varieties. Let p_{ij}: $X_1 \times X_2 \times X_3 \to X_i \times X_j$ be the obvious projections. Let $\Phi \subseteq X_1 \times X_2$ and $\Gamma \subseteq X_2 \times X_3$ be two irreducible cycles such that all the following maps are proper: $|\Phi| \to X_2$, $\Gamma \to X_3$, $p_{12}^{-1}(|\Phi|) \cap p_{23}^{-1}(|\Gamma|) \to X_1 \times X_3$, $p_{13}(p_{12}^{-1}(|\Phi|) \cap p_{23}^{-1}(|\Gamma|)) \to X_3$. In addition, assume that $p_{13}(p_{12}^{-1}(|\Phi|) \cap p_{23}^{-1}(|\Gamma|))$ has the expected dimension.

We can therefore define a refined cycle $\Gamma \circ \Phi$ in $Z_*(p_{13}(p_{12}^{-1}(|\Phi|) \cap p_{23}^{-1}(|\Gamma|)))$ and maps: Φ_*, Γ_*, and $(\Gamma \circ \Phi)_*$. We have $\Gamma_* \circ \Phi_* = (\Gamma \circ \Phi)_*$.

4.3. *Quotient Varieties*

The formalism recalled so far extends to the case of quotient varieties. See [8, Ex. 16.1.13]. In this case rational coefficients are necessary. The basic setup we need is as follows.

Z is a nonsingular integral variety; Z' is an integral variety; p_Z: $Z' \to Z$ is a surjective morphism; G is a finite group acting on Z and Z', compatibly with p_Z. There is a commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "p'"] \arrow[d, "p_Z"'] & Z'_1 \arrow[d, "p_{Z_1}"'] \\ Z \arrow[r, "p"] & Z_1 \end{tikzcd},$$

where $Z_1 = Z/G$, p is a quotient Galois morphism of degree $|G|$, $Z'_1 = Z'/G$, and p' is a quotient Galois morphism of degree $|G|$. Note that it follows that p and p' are separable. Define $[Z']$ and $[Z'_1]$ to be the corresponding fundamental classes.

We have that $p'^*[Z'_1] = [Z']$. We can define a Gysin-type map as follows,

$$\Phi_{[Z'_1]}\colon A_*(Z_1) \longrightarrow A_*(Z'_1), \qquad z_1 \longrightarrow \frac{1}{|G|} p'_*(p_Z^*(p^* z_1) \cap [Z']).$$

We need the following elementary fact, which can be checked directly using the definition of p^* for quotient maps; see [8, Ex. 1.7.6].

LEMMA 4.3.1. *Let R be a variety endowed with an action of a finite group G on it. Denote the quotient $p\colon R \to S$. Let $\sigma\colon A_*(R) \to A_*(R)$, $z \to \sum_{\gamma\in G} \gamma_* z$, be the so-called symmetrization operator. Then $\sigma = p^* p_*$.*

LEMMA 4.3.2. *We have the following commutative diagram*:

$$\begin{tikzcd}
A_*(Z') \arrow[r, "p'_*"] & A_*(Z'_1) \\
A_*(Z) \arrow[u, "\Phi_{[Z']}"] \arrow[r, "p_*"] & A_*(Z_1) \arrow[u, "\Phi_{[Z'_1]}"']
\end{tikzcd}.$$

Proof. For ease of notation, we denote a class $p_Z^*(a) \cap p'^* z'_1$, by $a \cdot p'^* z'_1$.

Let $z \in A_*(Z)$ and $z' \in A_*(Z')$. Define $\sigma\colon z \to \sum_{\gamma\in G} \gamma_* z$ and $\sigma'\colon z' \to \sum_{\gamma\in G} \gamma_* z'$. By virtue of Lemma 4.3.1, we have that $\sigma = p^* p_*$ and that $\sigma' = p'^* p'_*$. It follows that $p'^* p'_*(z \cdot p'^* z'_1) = \sigma'(z \cdot p'^* z'_1) = \sum_{\gamma\in G} \gamma_*(z \cdot p'^* z'_1) =$ (by the proper push-forward property) $= \sum_{\gamma\in G} (\gamma_* z \cdot \gamma_* p'^* z'_1) =$ (since $p'^* z'_1$ is G-invariant) $= \sum_{\gamma\in G} (\gamma_* z \cdot p'^* z'_1) = \sigma(z) \cdot p'^* z_1 = (p^* p_* z) \cdot p'^* z'_1 =$ (since $p^* p_* z$ is G-invariant) $= \frac{1}{|G|}[\sum_{\gamma\in G} \gamma_*(p^* p_* z)] \cdot p'^* z'_1 = \frac{1}{|G|} \sigma'(p^* p_* z) \cdot p'^* z'_1 = \frac{1}{|G|} p'^* p'_*(p^* p_* z \cdot p'^* z'_1)$. Since p'^* is injective, we have that $p'_*(z \cdot p'^* z'_1) = \frac{1}{|G|} p'_*(p^* p_* z \cdot p'^* z'_1)$, which is what we wanted to prove. ∎

4.4. *The Morphisms Γ_* and $\widehat{\Gamma}_*$*

Let X be an irreducible nonsingular quasi-projective surface defined over an algebraically closed field.

Recalling the notation introduced in Section 2, we summarize the results of this section by stating that the following diagram is commutative:

$$\begin{tikzcd}
A_*(\Gamma) = \oplus_\nu A_*(\Gamma^\nu) \arrow[r] & A_*(\widehat{\Gamma}) = \oplus_\nu A_*(\widehat{\Gamma}^\nu) \arrow[r] & A_*(X^{[n]}) \\
A_*(\mathscr{X}) = \oplus_\nu A_*(X^\nu) \arrow[u, "\Gamma_* = \oplus_\nu \Gamma^\nu_*"] \arrow[r] & A_*(\widehat{\mathscr{X}}) = \oplus_\nu A_*(X^\nu) \arrow[u, "\widehat{\Gamma}_* = \oplus_\nu \widehat{\Gamma}^\nu_*"] &
\end{tikzcd}.$$

By abuse of notation, the corresponding maps into $A_*(X^{[n]})$ will be denoted with the same symbol; e.g., $\Gamma_*\colon A_*(\mathscr{X}) \to A(X^{[n]})$ and $\widehat{\Gamma}_*\colon A_*(\widehat{\mathscr{X}}) \to A(X^{[n]})$.

5. THE CHOW GROUPS OF $X^{[n]}$

Let X be an irreducible nonsingular quasi-projective surface defined over an algebraically closed field. The reader should keep in mind Section 2.

5.1. *The Injectivity of* $\widehat{\Gamma}_*$

We shall freely use the formalism developed for correspondences over quotient varieties in Section 4.3.

LEMMA 5.1.1. *Let Z be an irreducible algebraic scheme of dimension n, quotient of a nonsingular algebraic scheme via a finite group. Let V and W be pure-dimensional cycles on Z of dimensions k and l. Let $f\colon Z \to Y$ be a proper morphism of algebraic schemes. If* $\dim f(|V| \cap |W|) < k+l-n$, *then* $f_*(V \cdot W) = 0$.

Proof. There is a well-defined refinement of the product in $A_{k+l-n}(|V| \cap |W|)$ which must map to zero. See [8, Ex. 8.3.12]. ∎

The following lemma is essentially a reformulation of [6]. Before stating it, we introduce some notation. Let ν be a partition of n. For $\underline{x} \in X^{(\nu)}_{reg}$, we set $F_\nu = (\pi^{-1}(\underline{x}))_{red}$, which we identify via P with $p_\nu^{-1}(\underline{x})$ (cf. Remark 2.0.2). We set $m_\nu := (-1)^{n-l(\nu)} \prod_{j=1}^{l(\nu)} \nu_j$. We denote by $[F_\nu] \cdot X^{[n]}_\nu \in A_0(F_\nu)$ the refined intersection defined by the closed imbeddings of F_ν and $X^{[n]}_\nu$ in $X^{[n]}_{\geq\nu}$.

LEMMA 5.1.2. $\deg([F_\nu] \cdot X^{[n]}_\nu) = m_\nu$.

Proof. The case $\nu = n^1$ is precisely the main result in [6]. The general case follows from the Künneth formula. ∎

In the following two propositions, we compute the compositions ${}^t\widehat{\Gamma}^\nu \circ \widehat{\Gamma}^\mu$. Here ${}^t-$ denotes, as usual, the transposed correspondence; see [8, Section 16.1].

PROPOSITION 5.1.3. *Let $\mu \neq \nu$ be two distinct partitions of n. Then ${}^t\widehat{\Gamma}^\nu \circ \widehat{\Gamma}^\mu = 0$ in $A_{l(\mu)+l(\nu)}(X^{(\mu)} \times X^{(\nu)})$.*

Proof. Consider $X^{(\mu)} \times X^{[n]} \times X^{(\nu)}$ together with the natural projections p, π, and q to $X^{(\mu)} \times X^{[n]}$, $X^{(\mu)} \times X^{(\nu)}$, and $X^{[n]} \times X^{(\nu)}$, respectively. By virtue of Lemma 5.1.1, it is enough to show that $\dim \pi(p^{-1}\widehat{\Gamma}^\mu \cap q^{-1}\, {}^t\widehat{\Gamma}^\nu) < l(\mu) + l(\nu)$.

One checks directly that $\dim \pi(p^{-1}\widehat{\Gamma}^\mu \cap q^{-1}\,{}^t\widehat{\Gamma}^\nu) = \dim(\overline{X}^{(n)}_\mu \cap \overline{X}^{(n)}_\mu)$. Since $\mu \neq \nu$, this last dimension is strictly less than $2\min(l(\mu), l(\nu))$. ∎

PROPOSITION 5.1.4. ${}^t\widehat{\Gamma}^\nu \circ \widehat{\Gamma}^\nu = m_\nu \Delta_{X^{(\nu)}}$ *in* $A_{2l(\nu)}(X^{(\nu)} \times X^{(\nu)})$.

Proof. The cycle ${}^t\widehat{\Gamma}^\nu \circ \widehat{\Gamma}^\nu$ is supported on the diagonal of $X^{(\nu)} \times X^{(\nu)}$, which is irreducible of the expected dimension; therefore, ${}^t\widehat{\Gamma}^\nu \circ \widehat{\Gamma}^\nu = c\Delta_{X^{(\nu)}}$, with $c \in \mathbb{Q}$. Let $\underline{x} \in X^{(\nu)}_{reg}$. Since the map $\widehat{\Gamma}^\nu_{reg} \to X^{(\nu)}_{reg}$ is flat, we have that $\widehat{\Gamma}^\nu_*([\underline{x}]) = [F_\nu]$. By virtue of Lemma 5.1.2 and of the projection formula, we have that

$$c = \deg\big({}^t\widehat{\Gamma}^\nu_* \circ \widehat{\Gamma}^\nu_*([\underline{x}])\big) = \deg\big({}^t\widehat{\Gamma}^\nu_*([F_\nu])\big) = \deg\big([F_\nu] \cdot X^{[n]}_\nu\big) = m_\nu.$$

∎

COROLLARY 5.1.5. *The natural map*

$$\widehat{\Gamma}_* = \bigoplus_{\nu \in \mathfrak{P}(n)} \widehat{\Gamma}^\nu_*\colon A_*(\widehat{\mathscr{X}}) = \bigoplus_{\nu \in \mathfrak{P}(n)} A_*\big(X^{(\nu)}\big) \longrightarrow A_*\big(X^{[n]}\big)$$

is injective.

Proof. By abuse of notation, denote by ${}^t\widehat{\Gamma}$ the correspondence $\coprod_\nu \frac{{}^t\widehat{\Gamma}^\nu}{m_\nu}$. Propositions 5.1.4 and 5.1.3 imply that ${}^t\widehat{\Gamma}_* \circ \widehat{\Gamma}_*$ is the identity. ∎

Remark 5.1.6. An argument identical to the one given above shows that Corollary 5.1.5 holds if we replace all spaces by the ones obtained by base change with respect to any open immersion $U \to X^{(n)}$.

5.2. *A Surjectivity Statement*

The following surjectivity statement is essential to proving Proposition 5.3.1, which, in turn, constitutes the surjectivity part of the main result of this paper, Theorem 5.4.1. The proof will be given at the end of this section, after a series of preparatory results.

PROPOSITION 5.2.1. *Let* $\mu \in \mathfrak{P}(n)$. *The map* $P_*(p^!_{\geq\mu}(-) \cap [\Gamma_{\geq\mu}])$: $A_*(\mathscr{X}_\mu) \to A_*(X^{[n]}_\mu)$ *is surjective.*

In order to prove Proposition 5.2.1, we need to make explicit the combinatorics involved. Let l be an integer, and let $\mathfrak{R}_l$ be the set of decompositions $\rho = \{I_1, \dots, I_r\}$ of the set $\{1, \dots, l\}$ into nonempty disjoint subsets; i.e., $\{1, \cdots, l\} = I_1 \coprod \cdots \coprod I_r$. For $\rho \in \mathfrak{R}_l$, define $\mathscr{D}_\rho = \{(x_1, \dots, x_l) \in X^l\colon x_i = x_j \Leftrightarrow i, j \in I_k$ for some $k\}$. For $\rho = \{1\}, \dots, \{l\}$, we have $\mathscr{D}_\rho = X^l_{reg} = X^l \backslash \mathit{Diagonals}$. For $\rho = \{1, \dots, l\}$, we have that $\mathscr{D}_\rho$ is the small diagonal.

Let $\mathfrak{P}_{\preceq\nu}(n) = \{\mu \in \mathfrak{P}(n)$ such that $\mu \preceq \nu\}$. Define $Q_\nu\colon \mathfrak{R}_{l(\nu)} \to \mathfrak{P}_{\preceq\nu}(n)$ by setting $Q_\nu(\{I_1, \dots, I_r\}) = (\sum_{i \in I_1} \nu_i, \dots, \sum_{i \in I_l} \nu_i)$. Note that this partition is not ordered; i.e., $\sum_{i \in I_1} \nu_i \not\geq \sum_{i \in I_2} \nu_i \cdots \not\geq \sum_{i \in I_l} \nu_i$, but see Remark 5.2.4.

Let $\nu\colon X^{(\nu)} \to \overline{X}^{[n]}_\nu$ be the already mentioned map defined by $\nu(x_1, \dots, x_{l(\nu)}) = \nu_1 x_1 + \cdots + \nu_{l(\nu)} x_{l(\nu)}$. Note that $\nu^{-1}(X^{(n)}_\nu) = X^\nu_{reg}$.

Note that $l(Q_\nu(\{I_1, \ldots, I_r\})) = r$. The proof of the following lemma is immediate:

LEMMA 5.2.2.

$$X^\nu \times_{X^{(n)}} X^{(n)}_\mu \neq \emptyset \Leftrightarrow \nu \succeq \mu \quad \textit{and} \quad \left(X^\nu \times_{X^{(n)}} X^{(n)}_\mu\right)_{red} = \coprod_{\rho \in Q_\nu^{-1}(\mu)} \mathcal{D}_\rho .$$

Let Γ^ν be the correspondences introduced in Section 2.

Remark 5.2.3. The group Σ_ν acts on $\mathfrak{R}_{l(\nu)}$ and Q_ν is Σ_ν-invariant. In particular, Σ_ν acts on $Q_\nu^{-1}(\mu)$.

Remark 5.2.4. For $\nu \in \mathfrak{P}(n)$ and $\rho = \{I_1, \ldots, I_r\} \in \mathfrak{R}_{l(\nu)}$, we can order the I_is so that we always have $\sum_{i \in I_1} \nu_i \geq \sum_{i \in I_2} \nu_i \cdots \geq \sum_{i \in I_r} \nu_i$.

Let $\rho \in \mathfrak{R}_{l(\nu)}$ and let $\mu := Q_\nu(\rho)$. We can identify $\mathcal{D}_\rho$ with X^μ_{reg} via the map $\delta_\rho \colon X^\mu_{reg} \to X^\nu$ sending the point $(y_1, \ldots, y_r) \in X^\mu_{reg}$ to $(x_1, \ldots, x_{l(\nu)})$ defined by $x_j = y_i$ precisely if $j \in I_i$. In this case $\nu \circ \delta_\rho = \mu_{|X^\mu_{reg}} \colon X^\mu_{reg} \to X^{(n)}$, hence $(\Gamma^\nu \times_{X^\nu} \mathcal{D}_\rho)_{red} = \Gamma^\mu_{|X^\mu_{reg}} =: \Gamma^\mu_{reg}$. These identifications will always be tacitly made in the sequel.

Remark 5.2.5. $p_\mu \colon \Gamma^\mu_{reg} \to X^\mu_{reg}$ is a Zariski locally trivial fibration. The fiber is isomorphic to the product of punctual Hilbert schemes $\mathcal{H}_{\mu_1} \times \cdots \times \mathcal{H}_{\mu_l}$ and it admits a cellular decomposition. See [9, Lemma 2.1.4] and [5].

Let (ν, ρ) be such that $Q_\nu(\rho) = \mu$. The pair (ν, ρ) defines a set of partitions $\nu^i \in \mathfrak{P}(\mu_i)$ as follows: if $\rho = \{I_1, \ldots, I_{l(\mu)}\}$, then $\mu_i = \sum_{j \in I_i} \nu_j$ and therefore $\nu^i = \{\nu_j\}_{j \in I_i}$ is a partition of μ_i.

DEFINITION 5.2.6. Define the open sets

$$U_\rho = \{(x_1, \ldots, x_{l(\nu)}) \in X^\nu, \text{ such that the subsets } \{x_j\}_{j \in I_i} \text{ are pairwise disjoint}\}.$$

In other words, we allow only points belonging to the same I_j to collapse; in particular, $U_\rho \supseteq \mathcal{D}_\rho$.

LEMMA 5.2.7. *Let (ν, ρ) be such that $Q_\nu(\rho) = \mu$, and let $\nu^i \in \mathfrak{P}(\mu_i)$ be the corresponding set of partitions. There is a canonical isomorphism*

$$\Gamma^\nu_{|U_\rho} = \prod \Gamma^{\nu^i}_{|U_\rho}.$$

Proof. Let $(x_1, \ldots, x_{l(\mu)}, \mathcal{I}) \in \Gamma^\nu_{|U_\rho}$. From the definition of U_ρ, it follows that $\mathcal{I}$ is the product of ideals $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_r$ of lengths $\mu_1, \mu_2, \ldots, \mu_r$ supported on $\{x_j\}_{j \in I_1}, \{x_j\}_{j \in I_2}, \ldots, \{x_j\}_{j \in I_r}$, whence the obviously bijective map from $\Gamma^\nu_{|U_\rho}$ to $\prod \Gamma^{\nu_i}_{|U_\rho}$. ∎

We have the following diagram,

$$\begin{tikzcd} \Gamma^{\mu}_{reg} \arrow[r] \arrow[d, "p_\mu"'] & \Gamma^{\nu}_{|U_\rho} \arrow[d, "p_\nu"] \\ \mathscr{D}_\rho = X^{\mu}_{reg} \arrow[r, "\delta_\rho"] & U_\rho \end{tikzcd},$$

which is cartesian modulo nilpotents (see Remark 4.1.3) and a map $\gamma_{\nu,\rho}\colon A_*(X^{\mu}_{reg}) \to A_*(\Gamma^{\mu}_{reg})$ defined by $\gamma_{\nu,\rho}(\beta) = p^!_\nu(\beta) \cap [\Gamma^\nu]$ (cf. Section 3).

LEMMA 5.2.8. *Let* $\alpha_{\nu,\rho} := \gamma_{\nu,\rho}([X^{\mu}_{reg}]) \in A_*(\Gamma^{\mu}_{reg})$ *and* $\beta \in A_*(X^{\mu}_{reg})$. *Then*

$$\gamma_{\nu,\rho}(\beta) = p^*_\mu(\beta) \cap \alpha_{\nu,\rho}.$$

Proof. It follows from the associativity property of refined products [8, Proposition 8.1.1.a]. ∎

What has been done so far can be summarized by the following:

LEMMA 5.2.9. *Let* μ *be a partition, and denote by* $\tilde{q}\colon \Gamma^{\mu}_{reg} \to X^{[n]}_{\mu}$ *the quotient map by the action of* Σ_μ. *Let* $W \subseteq A_*(\Gamma^{\mu}_{reg})$ *be the* $A_*(X^{\mu}_{reg})$-*submodule generated by the classes* $\{\alpha_{\nu,\rho}\}_{\rho\in Q_\nu^{-1}(\mu)}$. *The surjectivity of the map* $P_*(p^!_{\geq\mu}(-)\cap[\Gamma_{\geq\mu}])\colon A_*(\mathscr{X}_\mu) \to A_*(X^{[n]}_{\mu})$ *is equivalent to the surjectivity of the restriction of* $\tilde{q}_*\colon A_*(\Gamma^{\mu}_{reg}) \to A_*(X^{[n]}_{\mu})$ *to* W.

Proof. By virtue of Lemma 5.2.2,

$$\mathscr{X}_\mu = \coprod_{\nu\geq\mu} \coprod_{\rho\in Q_\nu^{-1}(\mu)} \mathscr{D}_\rho = \coprod_{\nu\geq\mu} \coprod_{\rho\in Q_\nu^{-1}(\mu)} X^{\mu}_{reg};$$

this last identification is made using the map described in Remark 5.2.4. The following diagram, where the equality $\sum \gamma_{\nu,\rho} = \sum p^*_\mu(\,) \cap \alpha_{\nu,\rho}$ is a consequence of Lemma 5.2.8, commutes,

$$\begin{tikzcd} A_*(\mathscr{X}_\mu) \arrow[rr, "P_*(p^!_{\geq\mu}()\cap[\Gamma_{\geq\mu}])"] \arrow[d] & & A_*(X^{[n]}_{\mu}) \\ \bigoplus_{\nu\geq\mu} \bigoplus_{\rho\in Q_\nu^{-1}(\mu)} A_*(X^{\mu}_{reg}) \arrow[rr, "\sum\gamma_{\nu,\rho}=\sum p^*_\mu()\cap\alpha_{\nu,\rho}"] & & A_*(\Gamma^{\mu}_{reg}) \arrow[u, "\tilde{q}*"'] \end{tikzcd},$$

whence the statement. ∎

We now study a special case which is crucial to the proof of Proposition 5.2.1. We consider $\mu' = n^1$. For every partition ν, there is only one $\delta_\rho = \delta\colon X \to X^{l(\nu)}$ and $p := p_{n^1}\colon \Gamma^{n^1} \to X$ is a Zariski locally trivial fibration, whose fiber is isomorphic to the length n punctual Hilbert scheme $\mathscr{H}_n$. For each $\nu \in \mathfrak{P}(n)$, we have $\alpha_\nu = p_\nu^!([X]) \cap [\Gamma^{n^1}]$ and the map $P\colon A_*(X)^{\oplus p(n)} \to A_*(\Gamma^{n^1})$ sending the collection $\{\beta_\nu\}$ to the class $\sum_\nu p^*(\beta_\nu) \cap \alpha_\nu$.

LEMMA 5.2.10. *The map P is surjective.*

Proof. By virtue of Lemma 3.0.2, it is enough to prove that the restrictions $\hat{\alpha}_\nu$ of the classes α_ν to any fiber generate $A_*(\mathscr{H}_n)$.

We may assume, without loss of generality, that X is projective, for the restrictions $\hat{\alpha}_\nu$ do not change. Let $i_x\colon \{point\} \to X$ be the imbedding of a point x. Note that $\hat{\alpha}_\nu = i_x^!([\Gamma^\nu])$. Let $g\colon \mathscr{H}_n \to X^{[n]}$ be the closed embedding and define $\hat{\beta}_\nu := g_* \hat{\alpha}_\nu \in A_*(X^{[n]})$.

Consider the pairing $A_*(X^{[n]}) \times A_*(X^{[n]}) \to \mathbb{Q}$ given by taking $(a, b) \to \deg a \cdot b = \int_{X^{[n]}} a \cdot b$, where the last product is the product in the Chow ring. Note that this pairing is almost never perfect. It descends to algebraic equivalence.

CLAIM. $\deg\big(\hat{\beta}_\nu \cdot [\overline{X}_\mu^{[n]}]\big) = 0$ *if and only if* $\mu \neq \nu$.

Proof of the Claim. Consider Γ^ν and $X^\nu \times \overline{X}_\nu^{[n]}$ as a pair of cycles in $X^\nu \times X^{[n]}$. They define two families of cycles on $X^{[n]}$: Γ_τ^ν and $\{X^\nu \times \overline{X}_\nu^{[n]}\}_\tau$, $\tau \in X^\nu$. See [8, Section 10]. In particular, we have $\Gamma_y^\nu = i_y^!([\Gamma^\nu])$. If $y \in X_{reg}^\nu$, then $\Gamma_y^\nu = [(p_\nu^{-1}(y))_{red}]$. Recalling that we have the canonical $\delta_\rho\colon X \to X^\nu$, we see that $\hat{\beta}_\nu = \Gamma_x^\nu$. In other words, Γ_y^ν and $\hat{\beta}_\nu$ belong to the same family of cycles. By construction, $\hat{\beta}_\nu$ can be represented, modulo algebraic equivalence, by a cycle supported at $(p_\nu^{-1}(y))_{red}$.

Note that $\hat{\beta}_\nu \cdot [\overline{X}_\mu^{[n]}] \in A_{l(\nu)-l(\mu)}(X^{[n]})$.

If $l(\nu) < l(\mu)$, or if $l(\nu) > l(\mu)$, then the degree is zero for trivial reasons.

If $l(\nu) = l(\mu)$, but $\mu \neq \nu$, then the degree is still zero. In fact, using refined intersection products in the context of algebraic equivalence, we can represent $\hat{\beta}_\nu$ as a cycle supported on a fiber $\pi^{-1}(y)$. This fiber does not meet $\overline{X}_\mu^{[n]}$.

Finally, if $\nu = \mu$, then, by virtue of [8, Corollary 10.1 and Proposition 10.2], we have that

$$\deg\big(\hat{\beta}_\nu \cdot [\overline{X}_\mu^{[n]}]\big) = \deg\big(\Gamma_y \cdot [\overline{X}_\mu^{[n]}]\big) = \deg\big([p_\nu^{-1}(y)] \cdot [\overline{X}_\nu^{[n]}]\big) = m_\nu.$$

The Claim follows easily.

By virtue of Corollary 5.1.5 (see also [4]), the classes $[\overline{X}_\mu^{[n]}]$ are independent. Jointly with the Claim just proved, this shows that the classes $\hat{\beta}_\nu$ are independent. It follows that the classes $\hat{\alpha}_\nu$ are independent. Their number is $p(n)$, which is equal to the dimension of $A_*(\mathcal{H}_n)$ (cf. [4], for example). This proves the lemma. ∎

Let us go back to the situation dealt with in Lemma 5.2.7. The diagram after that lemma and Lemma 5.2.10 imply immediately:

LEMMA 5.2.11. *The restriction of the cycle* $\alpha_{\nu,\rho} := p_\nu^!([\mathcal{D}_\rho]) \cap [\Gamma^\nu] \in A_*(\Gamma^\mu)$ *to a fiber of* p_μ *is the cycle* $\hat{\alpha}_{\nu^1} \times \cdots \times \hat{\alpha}_{\nu^r} \in A_*(\mathcal{H}_{\mu_1} \times \cdots \times \mathcal{H}_{\mu_r})$.

Let $\underline{\eta} = (\eta^1, \ldots, \eta^r)$ be a multipartition of μ. By this we mean that $\eta^i \in \mathfrak{P}(\mu_i)$. Set $l_i = l(\eta^i)$ and define the open sets $U_{\underline{\eta}} \subseteq \prod X^{\eta^i}$ by

$$U_{\underline{\eta}} = \left\{ \left(x_1^{(1)}, \ldots, x_{l_1}^{(1)}, \ldots, x_1^{(r)}, \ldots, x_{l_r}^{(r)}\right) \text{ such that } x_k^{(i)} \neq x_l^{(j)} \text{ if } i \neq j \right\}.$$

In other words, we allow collisions only among points of the same group. X_{reg}^μ can be identified with the closed subset of points in $U_{\underline{\eta}}$ such that $x_k^{(i)} = x_l^{(j)}$ if and only if $i = j$.

According to [8, 8.1.4], the diagram

$$\begin{array}{ccc}
\prod \Gamma^{\mu_i}_{|X^\mu_{reg}} & \longrightarrow & \prod \Gamma^{\eta^i} \\
\Big\downarrow {\scriptstyle \times p_{\mu_i}} & & \Big\downarrow {\scriptstyle \times p_{\eta^i}} \\
X^\mu_{reg} & \longrightarrow & \prod X^{\eta^i}
\end{array}$$

produces classes $\alpha_{\underline{\eta}} \in A_*(\prod \Gamma^{\mu_i}_{|X^\mu_{reg}})$, $\alpha_{\underline{\eta}} = (\times p_{\eta^i})^!([X^\mu_{reg}]) \cap [\prod \Gamma^{\eta^i}]$ whose restrictions to a fiber are the cycle $\hat{\alpha}_{\eta^1} \times \cdots \times \hat{\alpha}_{\eta^r} \in A_*(\mathcal{H}_{\mu_1} \times \cdots \times \mathcal{H}_{\mu_l})$.

LEMMA 5.2.12. *The set* $\{\alpha_{\underline{\eta}}\}$ *generates* $A_*(\Gamma^\mu_{reg})$ *as an* $A_*(X^\mu_{reg})$*-module.*

Proof. Since $\mathcal{H}_{\mu_1} \times \cdots \times \mathcal{H}_{\mu_r}$ has a cellular decomposition, $A_*(\mathcal{H}_{\mu_1} \times \cdots \times \mathcal{H}_{\mu_r}) = A_*(\mathcal{H}_{\mu_1}) \otimes \cdots \otimes A_*(\mathcal{H}_{\mu_r})$ has dimension $\prod p(\mu_i)$. The set in question is therefore a basis of this space, and the statement is a direct consequence of Lemma 3.0.2, Remark 5.2.5, and Lemma 5.2.11. ∎

Remark 5.2.13. We have already observed that a couple (ν, ρ) with $Q_\nu(\rho) = \mu$ determines a multipartition $\underline{\nu}_\rho$ of μ; Lemma 5.2.7 implies that $\alpha_{\underline{\nu}_\rho} = \alpha_{\nu,\rho}$.

Remark 5.2.14. The set $\{\alpha_{\nu,\rho}\}$ is only a subset of $\{\alpha_{\underline{\eta}}\}$, as shown by the following example.

EXAMPLE 5.2.15. Let $n = 4$, and let $\mu = 2^2$. There are four multipartitions: $(2, 2)$, $(2, 1^2)$, $(1^2, 2)$, $(1^2, 1^2)$, and four corresponding cycles which restrict to a basis for $A_*(\mathscr{H}_2 \times \mathscr{H}_2)$. The couples (ν, ρ) such that $Q_\nu(\rho) = 2^2$ are: if $\nu = 1^4$, then there are three different ρs which are conjugate by Σ_ν and give the same (0-dimensional) cycle corresponding to the multipartition $(1^2, 1^2)$; if $\nu = 2^2$, then there is only one ρ which gives the two-dimensional cycle corresponding to $(2, 2)$; if $\nu = 2 \cdot 1^2$, then the only ρ is $\{1\}, \{2, 3\}$, which induces the multipartition $(2, 1^2)$. This is consistent with the fact that $X^{2\cdot 1^2} \times_{X^{(4)}} X^{(4)}_{2^2}$ contains only one component $\mathscr{D}_\rho$ given by points of type $x_2 = x_3$. Note also that the map $\mathscr{D}_\rho \to X^{(4)}_{2^2}$ is 2:1, the quotient map by Σ_{2^2}.

This can be easily explained. Note first that the group Σ_μ acts on the set of multipartitions of μ. Given a multipartition $\underline{\eta} = (\eta^1, \dots, \eta^r)$ of μ, let $\eta^i = \eta^i_1 \geq \cdots \geq \eta^i_{l_i}$. The sequence of the η^i_j is a partition $\nu \in \mathfrak{P}(n)$; let $l = \sum l_i$ be its length. A permutation $\sigma \in \Sigma_l$, reordering the sequence in a non-increasing one, is identified up to left multiplication with Σ_ν. Define the subsets $I_i := \{\sigma((\sum_{k=0}^{i-1} l_k) + 1), \sigma((\sum_{k=0}^{i-1} l_k) + 2), \dots, \sigma(\sum_{k=0}^{i} l_k)\}$. We get $\{I_i\} \in \mathfrak{R}_l$, and $\sum_{k \in I_i} \nu_k = \mu_i$. This associates with $\underline{\eta}$ a couple (ν, ρ) such that $\rho \in Q_\nu^{-1}(\mu)$. A different choice of the permutation σ gives the same partition ν, whereas ρ is changed by the action of Σ_ν (cf. Remark 5.2.3). Starting from the couple (ν, ρ), instead, gives, as we have already observed, a multipartition $\hat{\eta}$ of μ. Note that this multipartition depends on the way the I_is were ordered. We thus have:

LEMMA 5.2.16. *$\hat{\eta}$ and $\underline{\eta}$ are in the same Σ_μ-orbit.*

Finally, let $q\colon \mathscr{D}_\rho = X^\mu_{reg} \to X^{(n)}_\mu$ and $\tilde{q}\colon \Gamma^\mu_{reg} \to X^{[n]}_\mu$ be the quotient maps by Σ_μ. It follows from [8, Ex. 1.7.6] that q^* (resp. $\tilde{q}^*$) identify $A_*(X^{(n)}_\mu)$ (resp. $A_*(X^{[n]}_\mu)$) with $A_*(\mathscr{D}_\rho)^{\Sigma_\mu}$ (resp. $A_*(\Gamma^\mu_{reg})^{\Sigma_\mu}$). By virtue of Lemma 4.3.1, if $\alpha \in A_*(\mathscr{D}_\rho)$, then $q^* q_* \alpha = \sum_{\sigma \in \Sigma_\mu} \sigma \cdot \alpha$. Similarly, if $\alpha \in A_*(\Gamma^\mu_{reg})$, then $\tilde{q}^* \tilde{q}_* \alpha = \sum_{\sigma \in \Sigma_\mu} \sigma \cdot \alpha$.

LEMMA 5.2.17. *The images under the map $\tilde{q}^*\tilde{q}_*$ of the $A_*(X^\mu_{reg})$-submodules generated by $\{\alpha_{\underline{\eta}}\}$ and $\{\alpha_{\nu,\rho}\}$ coincide.*

Proof. We fix our attention on a single Σ_μ-orbit. Set $\alpha_{\nu,\rho} = \alpha_0$. Let H denote the stabilizer of α_0 in Σ_μ, and choose a set $g_1 = e, g_2, \dots, g_r$ of representatives for Σ_μ/H, so that the orbit of α_0 is $\alpha_0, g_2\alpha_0, \dots, g_r\alpha_0$. Given a cycle $\alpha = \sum_{i=1}^r p^*(\beta_i) g_i \alpha_0$, let $\beta = (\sum_{i=1}^r g_i^{-1} p^*(\beta_i))\alpha_0$. Then $\tilde{q}^*\tilde{q}_*\alpha = \tilde{q}^*\tilde{q}_*\beta$. In fact, $\tilde{q}^*\tilde{q}_*\alpha = \sum_{g,j} g p^*(\beta_j) g g_j \alpha_0 = \sum_i g_i (\sum_{h \in H, j=1}^r h g_j^{-1} \times p^*(\beta_j)) g_i \alpha_0$, while $\tilde{q}^*\tilde{q}_*\beta = \sum_g g(\sum_j g_j^{-1} p^*(\beta_j)) g \alpha_0 = \sum_i (\sum_{h\in H, j=1}^r \times g_i h g_j^{-1} p^*(\beta_j)) g_i \alpha_0$. ∎

We are now in the position to prove Proposition 5.2.1. Lemmas 5.2.12 and 5.2.17 imply that the restriction of the push-forward map $\tilde{q}_*$ to the $A_*(X^{\mu}_{reg})$-submodule of $A_*(\Gamma^{\mu}_{reg})$ generated by the classes $\{\alpha_{\nu,\rho}\}$ is surjective. This implies Proposition 5.2.1 by virtue of Lemma 5.2.9.

5.3. *The Surjectivity of* $\widehat{\Gamma}_*$

Proposition 5.2.1 implies easily the following surjectivity result:

PROPOSITION 5.3.1. *Let* $[\Gamma]$ *be the fundamental cycle of* Γ. *The map*

$$P_*(p^*(-)\cap[\Gamma]) = \Gamma_*\colon A_*(\mathscr{X}) = \bigoplus_{\nu\in\mathfrak{P}(n)} A_*(X^{\nu}) \to A_*(X^{[n]})$$

is surjective.

Proof. By virtue of Lemma 4.1.2, we have, for every $\mu\in\mathfrak{P}(n)$, a commutative diagram with exact columns:

$$\begin{tikzcd}
A_*(\mathscr{X}_{\mu}) \arrow[r, "p^!_{\geq\mu}(-)\cap[\Gamma_{\geq\mu}]"] \arrow[d, "i_*"] & A_*(\Gamma_{\mu}) \arrow[r, "P_*"] \arrow[d] & A_*(X^{[n]}_{\mu}) \arrow[d] \\
A_*(\mathscr{X}_{\geq\mu}) \arrow[r, "p^*_{\geq\mu}(-)\cap[\Gamma_{\geq\mu}]"] \arrow[d, "j^*"] & A_*(\Gamma_{\geq\mu}) \arrow[r, "P_*"] \arrow[d] & A_*(X^{[n]}_{\geq\mu}) \arrow[d] \\
A_*(\mathscr{X}_{>\mu}) \arrow[r, "p^*_{>\mu}(-)\cap[\Gamma_{>\mu}]"] \arrow[d] & A_*(\Gamma_{>\mu}) \arrow[r, "P_*"] \arrow[d] & A_*(X^{[n]}_{>\mu}) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}.$$

We proceed by decreasing induction on μ, proving that $P_*(p^*(-)\cap[\Gamma_{\geq\mu}])\colon A_*(\mathscr{X}_{\geq\mu})\to A_*(X^{[n]}_{\geq\mu})$ is surjective. The statement in the case $\mu = 1^n$ is clearly true since $X^{[n]}_{1^n} = X^{(n)}_{1^n} = \mathscr{X}_{1^n}/\Sigma_n$. Suppose now that the surjectivity of $P_*(p^*(-)\cap[\Gamma_{>\mu}])\colon A_*(\mathscr{X}_{>\mu})\to A_*(X^{[n]}_{>\mu})$ has been established. By virtue of Proposition 5.2.1, the map in the first line of the diagram is surjective, whence the surjectivity of the second line and the statement. ∎

COROLLARY 5.3.2. *The natural map*

$$\widehat{\Gamma}_*\colon A_*(\widehat{\mathscr{X}}) = \bigoplus_{\nu\in\mathfrak{P}(n)} A_*(X^{(\nu)}) \longrightarrow A_*(X^{[n]})$$

is surjective.

Proof. Immediate from Proposition 5.3.1 and Section 4.4.

Remark 5.3.3. Corollary 5.3.2 holds if we replace all spaces by the ones obtained by base change with respect to any open immersion $U \to X^{(n)}$. In fact, it is sufficient to use the corollary together with the standard exact sequence [8, Proposition 1.8].

5.4. *The Isomorphism of Chow Groups*

The main result of this paper now follows easily. It does *not* hold with $\mathbb{Z}$ coefficients.

THEOREM 5.4.1. *Let X be an irreducible nonsingular algebraic surface defined over an algebraically closed field. The natural map*

$$\widehat{\Gamma}_* = \bigoplus_{\nu\in\mathfrak{P}(n)} \widehat{\Gamma}^{\nu}_*\colon A_*(\widehat{\mathscr{X}}) = \bigoplus_{\nu\in\mathfrak{P}(n)} A_*(X^{(\nu)}) \longrightarrow A_*(X^{[n]})$$

is an isomorphism.

Proof. Injectivity and surjectivity are proved in Corollaries 5.1.5 and 5.3.2, respectively. ∎

The following is an immediate consequence of Theorem 5.4.1. Let Y be a scheme. We denote by $K_o(Y)$ the Grothendieck group of coherent sheaves on Y and we define $K_o(Y)_{\mathbb{Q}} := K_o(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$.

COROLLARY 5.4.2. *There is a natural isomorphism of Grothendieck groups with $\mathbb{Q}$-coefficients*

$$\widehat{\Gamma}_* = \bigoplus_{\nu\in\mathfrak{P}(n)} \widehat{\Gamma}^{\nu}_*\colon K_o(\widehat{\mathscr{X}})_{\mathbb{Q}} = \bigoplus_{\nu\in\mathfrak{P}(n)} K_o(X^{(\nu)})_{\mathbb{Q}} \longrightarrow K_o(X^{[n]})_{\mathbb{Q}}.$$

Proof. It follows immediately from Theorem 5.4.1 and [8, Corollary 18.3.2]. ∎

Remark 5.4.3. Corollary 5.4.2 and the localization theorem for equivariant K-theory [19] imply that one has an isomorphism $K_o^{\Sigma_n}(X^n)_{\mathbb{Q}} \simeq K_o(X^{[n]})_{\mathbb{Q}}$. We expect similar statements to hold for the higher K-theory. M. Haiman informed us, after this paper was written, that he has proved the $n!$ conjecture in his paper [13]. Coupled with [13], the paper [2] implies, in characteristic zero, a statement for the equivariant derived category analogue to the one for equivariant K-theory and a weaker version of Theorem 5.4.1. M. Haiman has also informed us that [13] leads to a geometrically explicit version of Corollary 5.4.2.

Remark 5.4.4. Theorem 5.4.1 and Corollary 5.4.2 hold if we replace $\mathscr{X}$ and $X^{[n]}$ by the corresponding open sets obtained by base change with respect to any open immersion $U \to X^{(n)}$. See Remarks 5.1.6 and 5.3.3.

Remark 5.4.5. Note that, by virtue of [8, Ex. 1.7.6], $A_*(X^{(\nu)}) = A_*(X^\nu)^{\Sigma_\nu}$. However, unless X itself has a cellular decomposition, the natural map $A_*(X)^{\otimes l(\nu)} \to A_*(X^{(\nu)})$ is not necessarily surjective. This contrasts with singular cohomology, where one has the Künneth formula. In particular, Nakajima–Grojnowski's constructions will detect only a proper subspace of $\oplus_{n\geq 0} A_*(X^{[n]})$.

Remark 5.4.6. With minor modifications, which we leave to the interested reader, Theorem 5.4.1 holds (1) for a geometrically irreducible smooth surface X defined over any field, and (2) in the analytic context where X is a smooth complex analytic surface (see [8, Ex. 19.2.5]).

6. THE MOTIVE OF $X^{[n]}$

Let X be an irreducible nonsingular quasi-projective surface defined over an algebraically closed field.

6.1. *The Correspondences* Δ_ν

We freely use refined products over quotient varieties and the formalism of correspondences between quotient varieties together with their standard properties. See Section 4. Let $\nu \in \mathfrak{P}(n)$ and consider $(X_\nu^{[n]} \times_{X^{(n)}} X_\nu^{[n]})_{red}$. Being the quotient of a Zariski locally trivial fibration with irreducible fiber by the action of a finite group, this space is a $2n$-dimensional irreducible locally closed subset of $X^{[n]} \times X^{[n]}$. Let $D^\nu \subseteq X^{[n]} \times X^{[n]}$ be its closure:

$$D^\nu := \overline{\left\{(a, b) \in X^{[n]} \times X^{[n]} \mid \pi(a) = \pi(b) \in X_\nu^{(n)}\right\}} \in Z_{2n}(X^{[n]} \times X^{[n]}).$$

Note that the image of D^ν under either projection is the closed stratum $\overline{X}_\nu^{[n]}$.

Remark 6.1.1. D^ν is an irreducible component of $(\overline{X}_\nu^{[n]} \times_{X^{(n)}} \overline{X}_\nu^{[n]})_{red}$. The latter contains all the D^μ such that $\mu \preceq \nu$ precisely as its irreducible components.

Let $\nu \in \mathfrak{P}(n)$. The components of $p_{12}^{-1}({}^t\widehat{\Gamma}^\nu) \cap p_{23}^{-1}(\widehat{\Gamma}^\nu)$ in $X^{[n]} \times X^{(\nu)} \times X^{[n]}$ are all of the expected dimension. The refined formalism mentioned above, Remark 3.0.1, and Remark 6.1.1 show that the following is a well-defined $2n$-dimensional cycle with rational coefficients,

$$\Delta_\nu := \frac{1}{m_\nu} \widehat{\Gamma}^\nu \circ {}^t\widehat{\Gamma}^\nu \in Z_{2n}(X^{[n]} \times X^{[n]}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

with the property that Δ_ν is supported precisely on $(\overline{X}_\nu^{[n]} \times_{X^{(n)}} \overline{X}_\nu^{[n]})_{red}$ and that

$$\Delta_\nu = \sum_{\nu \succeq \nu'} \epsilon^\nu_{\nu'} D^{\nu'},$$

where the numbers $\epsilon^\nu_{\nu'}$ are nonzero rational numbers with the same sign as m_ν.

LEMMA 6.1.2. *Let ν and μ be partitions of n. Then*

$$\Delta_\nu \circ \Delta_\mu = \delta_{\nu\mu}\Delta_\nu \in Z_{2n}(X^{[n]} \times X^{[n]}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\delta_{\nu\mu}$ is the usual Krönecker function.

In particular, the Δ_ν are mutually orthogonal projectors and

$$\Delta_{\nu *}\colon A_*(X^{[n]}) \to A_*(X^{[n]}), \qquad \Delta_{\nu *} \circ \Delta_{\mu_*} = \delta_{\mu\nu}\Delta_{\nu *}.$$

Proof. By the associativity of the composition of correspondences, we have

$$\Delta_\nu \circ \Delta_\mu = \frac{\widehat{\Gamma}^\nu \circ {}^t\widehat{\Gamma}^\nu}{m_\nu} \circ \frac{\widehat{\Gamma}^\mu \circ {}^t\widehat{\Gamma}^\mu}{m_\mu} = \frac{\widehat{\Gamma}^\nu}{m_\nu} \circ \left(\frac{{}^t\widehat{\Gamma}^\nu \circ \widehat{\Gamma}^\mu}{m_\mu}\right) \circ {}^t\widehat{\Gamma}^\mu.$$

We conclude by Proposition 5.1.3 and Proposition 5.1.4. ∎

Recall that F_μ denotes the reduced fiber $(\pi^{-1}(\underline{x}))_{red}$, where $\underline{x} \in X^{(n)}_\mu$.

LEMMA 6.1.3. *Let ν and μ be partitions of n.*

(i) *If $\nu \not\succeq \mu$, then $\Delta_{\nu *}([F_\mu]) = 0$. If $\nu = \mu$, then $\Delta_{\nu *}([F_\mu]) = [F_\mu]$.*

(ii) *If $\nu \not\succeq \mu$, then $D_{\nu *}([F_\mu]) = 0$. If $\nu = \mu$, then $\Delta_{\nu *}([F_\mu]) = c_\nu[F_\nu]$, where $\mathbb{Q} \ni c_\nu \neq 0$.*

Proof. We compute $\Delta_{\nu *}$ using refined intersections so that the classes $[F_\mu]$, which are nonzero in $A_*(F_\mu)$, may be zero in $A_*(X^{[n]})$. Of course, this does not happen if, for example, X is projective.

If ν does not refine μ, then $p^{-1}(F_\mu) \cap \Delta_\nu = \varnothing$, and the first parts of (i) and (ii) follow at once. Note that since we have proved the first parts of (i) and (ii), (i) implies (ii) by consideration of supports. We now prove the second part of (i). Let $\mu = \nu$. Note that $\Delta_{\nu *}([F_\nu]) = (1/m_\nu)\widehat{\Gamma}^\nu_*({}^t\widehat{\Gamma}^\nu_*([F_\nu]))$. The zero cycle ${}^t\widehat{\Gamma}^\nu_*([F_\nu])$ is supported at the point $\nu^{-1}(\underline{x}) \in X^{(\nu)}$. By virtue of the projection formula, its degree equals $\deg([F_\nu] \cdot [\overline{X}_\nu^{[n]}]) = m_\nu$; see Lemma 5.1.2. In other words, ${}^t\widehat{\Gamma}^\nu_*([F_\nu]) = m_\nu[\nu^{-1}(\underline{x})]$. We apply $\widehat{\Gamma}^\nu_*$ and we find the conclusion. ∎

Note that the diagonal $\Delta = D^{1^n}$. The main reason for introducing the correspondences Δ_ν is the following.

PROPOSITION 6.1.4. *The map* $(\sum_{\nu\in\mathfrak{P}(n)}\Delta_\nu - \Delta)_*$ *is the zero map. In particular,* $\sum_\nu \Delta_{\nu *} = \Delta_* = Id_{A_*(X^{[n]})}$.

Proof. This follows immediately from Theorem 5.4.1, which implies that ${}^t\widehat{\Gamma}_*$ is the inverse of $\widehat{\Gamma}_*$, and from the fact that $\widehat{\Gamma}_* \circ {}^t\widehat{\Gamma}_* = \sum_{\nu\in\mathfrak{P}(n)}\Delta_{\nu *}$. ∎

PROPOSITION 6.1.5. $\sum_\nu \Delta_\nu - \Delta = 0$ *in* $Z_{2n}(X^{[n]}\times X^{[n]})\otimes_{\mathbb{Z}}\mathbb{Q}$.

Proof. Without loss of generality, we may assume that X is projective. Note that, for every $\nu \neq 1^n$, D^ν_* is identically zero on $A_0(X^{[n]})$. It follows, by virtue of Lemma 6.1.3, that we can write $\sum_{\nu\in\mathfrak{P}(n)}\Delta_\nu - \Delta = \sum_{\nu\neq 1^n} p_\nu D^\nu$, where p_ν are suitable rational numbers. Seeking a contradiction, assume that there is at least one partition μ for which $p_\mu \neq 0$. Choose a partition θ which is maximal, with respect to the partial ordering on partitions, among all partitions μ for which $p_\mu \neq 0$. We have, again by virtue of Lemma 6.1.3, that

$$0 = \left(\sum_\nu p_\nu D^\nu\right)_* [F_\theta] = p_\theta D^\theta_*([F_\theta]) \neq 0.$$

The contradiction stems from the projectivity assumption, for then no effective cycle can be trivial. ∎

6.2. *The Structure and the Generating Function of Motives*

Let X be an irreducible nonsingular projective surface defined over an algebraically closed field. The correspondence $\widehat{\Gamma}^\nu$ defines a morphism, which by abuse of notation we denote by the same symbol, of effective Chow motives with rational coefficients:

$$\widehat{\Gamma}^\nu\colon \big(X^{(\nu)}, \Delta_{X^{(\nu)}}\big)(n-l(\nu)) \longrightarrow \big(X^{[n]}, \Delta_{X^{[n]}}\big).$$

See [15, p. 459]. The Tate-type shift is in accordance with the usual convention $(\mathbb{P}^1, \Delta_{\mathbb{P}^1})(1) = (\mathbb{P}^1, p)$, where $p = \mathbb{P}^1 \times \{0\}$. This morphism admits, in the category of Chow motives with rational coefficients, a right inverse. This is given, again by abuse of notation, by ${}^t\widehat{\Gamma}^\nu$. By virtue of [15, p. 453], we can split-off a direct summand:

$$\widehat{\Gamma}^\nu\colon \big(X^{(\nu)}, \Delta_{X^{(\nu)}}\big)(n-l(\nu)) \simeq \big(X^{[n]}, \Delta_\nu\big).$$

By virtue of Lemma 6.1.2, the projectors Δ_ν are mutually orthogonal, so that we can split-off a direct summand for each partition:

$$\widehat{\Gamma} := \bigoplus_{\nu\in\mathfrak{P}(n)} \widehat{\Gamma}^\nu\colon \bigoplus_{\nu\in\mathfrak{P}(n)} \big(X^{(\nu)}, \Delta_{X^{(\nu)}}\big)(n-l(\nu))$$

$$\simeq \bigoplus_{\nu\in\mathfrak{P}(n)} \big(X^{[n]}, \Delta_\nu\big) = \left(X^{[n]}, \sum_{\nu\in\mathfrak{P}(n)} \Delta_\nu\right).$$

The following is the main theorem of this section and follows immediately from above and Proposition 6.1.5.

THEOREM 6.2.1. *Let X be an irreducible nonsingular projective surface defined over an algebraically closed field. There is a natural isomorphism of effective Chow motives with rational coefficients*

$$\left(X^{[n]}, \Delta_{X^{[n]}}\right) \simeq \bigoplus_{\nu \in \mathfrak{P}(n)} \left(X^{(\nu)}, \Delta_{X^{(\nu)}}\right)(n - l(\nu)).$$

Remark 6.2.2. Theorem 6.2.1 and Theorem 6.2.5 below hold over any field. We leave the necessary but easy modifications to the reader.

Remark 6.2.3. Over the complex numbers, Theorem 6.2.1 gives immediately the structure of the singular cohomology $H^*(X^{[n]}, \mathbb{Q})$, together with its Hodge structure. The resulting isomorphisms coincide with the ones obtained in [4].

Remark 6.2.4. After this paper was written, L. Göttsche informed us that he has calculated, by a different method [11], the class of $X^{[n]}$ in the Grothendieck ring of varieties over an algebraically closed field of characteristic zero. As a consequence, he calculates the class of the motive of $X^{[n]}$ in the Grothendieck ring of Chow motives. This latter invariant is strictly coarser than Chow motives: any relation between Chow motives reflects directly in the corresponding Grothendieck ring, but *not* vice versa. The main result of [11] is implied by Theorem 6.2.1, which, in addition, is valid over any field. The paper [11] also proves that the standard conjectures on cycles would imply, for an algebraically closed field of characteristic zero, a structure theorem for the Chow groups with rational coefficients of $X^{[n]}$; this would prove a slightly weaker version of Theorem 5.4.1. The methods of [11] apply to some other spaces as well.

Voevodsky [20] has defined several motivic categories over a field k: $DM_{gm}^{eff}(k)$ (cf. [20, 2.1.1]), $DM_{-}^{eff}(k)$ (cf. [20, 3.1.12]), $DM_{-,et}^{eff}(k)$ (cf. [20, 3.3]), $DM_h(k)_{\mathbb{Q}}$ (cf. [20, 21]). These are, essentially, categories of bounded complexes of pre-sheaves on certain sites. There is an obvious formalism of shifts of complexes and a formalism of Tate-type shifts. In addition, there are natural transformations between the category of effective Chow motives with rational coefficients and these other categories (see [20, 2.1.4]):

$$Chow^{eff}(k)_{\mathbb{Q}} \to DM_{gm}^{eff}(k) \to DM_{-}^{eff}(k) \to DM_{-,et}^{eff}(k) \to DM_h(k)_{\mathbb{Q}}.$$

The formalism holds for quotient varieties.

Let us denote by $M(Z)$ the object in any of the above categories corresponding to a variety Z. Theorem 6.2.1 and the natural transformations mentioned above imply the following.

THEOREM 6.2.5. *Let X be an irreducible nonsingular projective surface defined over an algebraically closed field. Then*

$$M(X^{[n]}) \simeq \bigoplus_{\nu \in \mathfrak{P}(n)} M(X^{(\nu)})(n - l(\nu))[2n - 2l(\nu)].$$

This is formally reminiscent of (1) the decomposition theorem for the Douady–Barlet morphism [4], where shifts for complexes of sheaves appear, and (2) the Hodge structure of $X^{[n]}$ (see [4], for example), where Tate shifts appear. It is by staring at those two formulas that we became convinced that the statement of Theorem 6.2.5 should hold. Theorem 5.4.1 and Theorem 6.2.1 were the means to realize this isomorphism.

It is amusing to note that there is a generating function for this picture, the one that generates partitions. In fact, skipping the book-keeping details, the formula is as follows,

$$\sum_{n=0}^{\infty} X^{[n]}(-n)[-2n] = \prod_{m=1}^{\infty} \frac{1}{(1 - X_m(-1)[-2]q^m)},$$

where it is understood that in the right-hand side we identify monomials of type $X_1^{a_1} \cdots X_t^{a_t}$ with $X^{(\nu)}$, where $\nu = 1^{a_1} \cdots t^{a_t} \in \mathfrak{P}(t)$.

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