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# The Douady Space of a Complex Surface

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We prove that a standard realization of the direct image complex via the so-called Douady–Barlet (Hilbert-Chow in the algebraic case) morphism associated with a smooth complex analytic surface admits a natural decomposition in the form of an injective quasi-isomorphism of complexes. This is a more precise form of a special case of the decomposition theorems of Beilinson, Bernstein, Deligne, Gabber, and M. Saito of which the proof we present is independent; in addition it is elementary and transparent and does not use either perverse sheaves, the methods of positive characteristic, nor Saito's theory of Mixed Hodge Modules. The proof hinges on the special case of the bi-disk in the complex affine plane where we make explicit use of a construction of Nakajima's and of the corresponding representation-theoretic interpretation foreseen by Vafa and Witten. Some consequences of the decomposition theorem: the Göttsche Formula holds for complex surfaces; interpretation of the rational cohomologies of Douady spaces as a kind of Fock space; new proofs of results of Briançon and Ellingsrud and Stromme on punctual Hilbert schemes; and computation of the mixed Hodge structure of the Douady spaces in the Kähler case. We also derive a natural connection with Equivariant $K$-Theory for which, in the case of algebraic surfaces, Bezrukavnikov and Ginzburg have proposed a different approach. 

*Contents.*



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## INTRODUCTION

Let $X$ be a smooth complex analytic surface, $n$ be a non-negative integer, and $X^{[n]}$ be the Douady space of zero-dimensional analytic subspaces of $X$ of length $n$. The spaces $X^{[n]}$ are $2n$-dimensional complex manifolds. If $X$ is algebraic, then they are the usual Hilbert schemes. These manifolds have been intensively studied from a "local" perspective; see [6, 10, 12, 14, 21–23]; this list is by no means complete.

Let $X$ be algebraic. The Göttsche Formula (see Theorem 5.2.1(2)) gives the Betti numbers of $X^{[n]}$ in terms of the ones of $X$. Vafa and Witten [32] remarked that the Göttsche Formula is the character formula of the standard *irreducible* representation of a certain infinite dimensional super Lie algebra $\mathscr{H}(X)$, called the Heisenberg/Clifford algebra, modeled on the rational cohomology $H^*(X)$ of $X$. As a consequence, the vector space $\mathbb{H}(X) := \bigoplus_{n \geqslant 0} H^*(X^{[n]})$ can be seen, abstractly, as an *irreducible* highest weight $\mathscr{H}(X)$-module.

Motivated by this remark, Nakajima [27] realized *geometrically*, and for every complex surface $X$, the space $\mathbb{H}(X)$ as a highest weight $\mathscr{H}(X)$-module by introducing certain operators acting on $\mathbb{H}(X)$ induced by correspondences in the products of Douady spaces. Grojnowski has announced similar results [18]. In this context, the Göttsche Formula Theorem 5.2.1(2) becomes the statement that if $X$ is algebraic, then the geometrically realized action of $\mathscr{H}(X)$ on $\mathbb{H}(X)$ is irreducible. In particular, $\mathbb{H}(X)$ can be re-built, at least in principle, from $H^*(X)$ by means of explicit geometric operators.

This new and beautiful structure of the Hilbert schemes $X^{[n]}$ ($X$ algebraic) emerges because all values of $n$ have been considered *simultaneously*.

All known proofs of the Göttsche Formula, for $X$ algebraic, rely, to start with, on results of Briançon [6] and Ellingsrud and Stromme [10] on punctual Hilbert schemes. Each individual proof then relies on either Deligne's solution to the Weil Conjectures [16], on the Beilinson–Bernstein–Deligne–Gabber Decomposition Theorem for perverse sheaves [17], or on Deligne's theory of mixed Hodge structures [7].

In this paper we develop a new approach by taking the $\mathscr{H}(X)$-action on $\mathbb{H}(X)$ as a starting point.

Our approach, inspired by Nakajima [27], but otherwise self-contained, underlines the importance of the representation-theoretic side, gives an elementary proof of the relevant decomposition theorem and of the Göttsche Formula, and it extends the theory to the complex analytic case.

In particular, *all* the results mentioned above, which hold in the algebraic case, are proved here to hold, more generally, in the analytic context. They are all consequences of our analysis of the special case $X = \varDelta$ the bi-disk in $\mathbb{C}^2$ and of our Decomposition Theorem 4.1.1. Moreover, we establish a new, natural relation with Equivariant $K$-Theory.

This analytic viewpoint would seem to be more natural.

Nakajima [27] asks whether it is possible to extend the picture drawn in the algebraic case to differentiable four-manifolds and in this paper we show that this can be done for complex analytic surfaces.

The analysis builds on the special case $X = \varDelta$. Here the role of the usual Heisenberg algebra $\mathscr{H}(\varDelta)$ is paramount: we determine an explicit, natural and geometrically meaningful basis for $\mathbb{H}(\varDelta)$. The case $X = \mathbb{C}^2$ is analogous and we recover results in [10] with the basic difference that our basis differs from the one in [11]. We exploit the geometric meaning of our basis for $\mathbb{H}(\varDelta)$ and prove, *for every* complex analytic surface $X$, a precise form of the Decomposition Theorem for the Douady–Barlet morphism $\pi\colon X^{[n]} \to X^{(n)}$ in the form of an injective quasi-isomorphism of complexes. The relevant morphism in [4], used in [17] to prove the Göttsche Formula, is in a derived category.

The Göttsche Formula and the irreducibility of $\mathbb{H}(X)$ follow formally, and so does the determination, first obtained in [17] using Saito's theory of mixed Hodge modules, of the mixed Hodge structure of $X^{[n]}$ in the algebraic (or "Kähler") case. We also re-obtain results in [6, 10] on punctual Hilbert schemes. Another easy consequence of our Decomposition Theorem is the construction of a natural additive isomorphism between the rational $\mathfrak{S}_n$-Equivariant $K$-Theory of $X^n$ (here $\mathfrak{S}_n$ acts on $X^n$ by permuting the factors) and the rational $K$-Theory of $X^{[n]}$ : $K_{\mathfrak{S}_n}(X^n) \otimes \mathbb{Q} \simeq K(X^{[n]}) \otimes \mathbb{Q}$. Bezrukavnikov and Ginzburg [5] have proposed to construct, in the algebraic case, a different natural map. Our motivation and proof are different from theirs.

The paper is organized as follows. In Section 2 we give a new, explicit, construction of $X^{[n]}$ and of the Douady–Barlet morphism $\pi\colon X^{[n]} \to X^{(n)}$. The building blocks are the Douady spaces for the bi-disk $\Delta^{[m]}$, $m \leqslant n$. These, in turn, are constructed using the "toy model" of Nakajima [28]. For $X = \Delta$, $\mathbb{C}^2$, we use the $\mathscr{H}(X)$-action on $\mathbb{H}(X)$ to compute the Betti numbers of $X^{[n]}$ and determine a canonical basis for $\mathbb{H}(X)$, Theorem 2.5.1. We use this result to re-prove the Ellingsrud–Stromme formula for the Betti numbers of punctual Hilbert schemes of a surface, Corollary 2.6.1, and the irreducibility result of Briançon, Corollary 2.6.2.

Section 3 is preparatory for the Decomposition Theorem. We study the Douady–Barlet morphism and its natural stratification. The normalizations of the closures of the strata in $X^{(n)}$ play a basic role. This section discusses the natural identifications which we obtain between objects on these normalizations, on $X^{(n)}$ and on $X^{[n]}$. We then define a certain injective morphism of complexes $\Psi$; see Proposition 3.6.2.

In Subsection 4.1 we prove our Decomposition Theorem, Theorem 4.1.1: $\Psi$ is a quasi-isomorphism. In Section 5 we deduce formal consequences of Theorem 4.1.1 which are new in the analytic case: Corollary 5.1.1 ($R^q\pi_*\mathbb{Q}_{X^{[n]}}$), Theorem 5.1.2 (degeneration of the Leray spectral sequence), Theorem 5.2.1 (Göttsche Formula), Corollary 5.2.3 (Euler numbers of $X^{[n]}$), Theorem 5.2.4 (irreducibility of $\mathbb{H}(X)$ as a highest weight $\mathscr{H}(X)$-module), Theorem 5.3.1 (mixed Hodge structure of $X^{[n]}$). Finally we prove Theorem 5.4.3 ($K_{\mathfrak{S}_n}(X^n) \otimes \mathbb{Q} \simeq K(X^{[n]}) \otimes \mathbb{Q}$).

## 1. NOTATION AND TERMINOLOGY

In this paper, the term *complex surface* refers to a smooth, connected, complex-analytic two dimensional manifold with countable topology.

— $D'^p$ the sheaf of $p$-currents on a smooth manifold $M$; if the topology is second countable, then the "de Rham" complex $\mathbb{C}_M \to D'^\bullet$ is a fine resolution of $\mathbb{C}_M$. This is the only reason why we require the topology of $X$ to be countable.

— $\Delta := \{\, (z_1, z_2) \subset \mathbb{C}^2,\ \ |z_i| < 1,\ \ i = 1, 2 \,\}$, the unit bi-disk.

— $X$ a complex surface.

— $\mathfrak{S}_n$ the symmetric group over $n$ elements.

— $P(n)$ the partitions of $n$. We use two standard pieces of notation. $\tilde{\nu} := (\nu_1, ..., \nu_k)$, with $\nu_j > 0$ and $\sum_{j=1}^k \nu_j = n$. The same partition can be represented as follows: (a-notation) $\mathbf{a} = \mathbf{a}(\tilde{\nu}) = (a_1, ..., a_n)$, where $a_i$ is the number of times that the number $i$ appears in the partition $\tilde{\nu}$. Note that $\sum_{i=1}^n ia_i = n$.

— $\lambda(\tilde{\nu}) = \lambda(\mathbf{a}) = k = \sum_{i=1}^{n} a_i$ the length of the partition.

— $X^n$, $X^{(n)}$ and $X^{[n]}$ the cartesian product, the symmetric product, and the Douady space of zero-dimensional analytic subspaces of $X$ of length $n$.

— $\pi = \pi_n \colon X^{[n]} \to X^{(n)}$, the natural Douady–Barlet morphism.

— $X^{(n)}_{(\tilde{\nu})}$ or $X^{(n)}_{(\mathbf{a})}$ the locally closed smooth subspaces of $X^{(n)}$ locus of points of the form $\sum_{j=1}^{k} \nu_i x_j$, where the $x_j$ are pairwise distinct.

— $X^{[n]}_{(\tilde{\nu})}$ (or $X^{[n]}_{(\mathbf{a})}$) $:= \pi^{-1}(X^{(n)}_{(\tilde{\nu})})$, where the pre-image is taken with the induced reduced structure.

— $\Delta^{[n]}_o := \pi^{-1}(no)$, the closed reduced analytic subspace of $\Delta^{[n]}$ locus of subspaces of $\Delta$ of length $n$ supported at the origin $o \in \Delta$ (the so-called *punctual Hilbert schemes*).

— $X^{(\mathbf{a})}$, $\mathbf{a} \in P(n)$, the spaces $\prod_{i=1}^{n} X^{(a_i)}$.

— $K_{\mathbf{a}} \colon X^{(\mathbf{a})} \to \overline{X^{(n)}_{(\mathbf{a})}}$ the natural map sending

$$(x_1^1 + \cdots + x_{a_1}^1, \ldots, x_1^n + \cdots + x_{a_n}^n) \longrightarrow \sum_{i=1}^{n} i(x_1^i + \cdots + x_{a_i}^i).$$

$$X^{(n)}_l := \overline{\coprod_{\lambda(\mathbf{a})=l} X^{(n)}_{(\mathbf{a})}} = \coprod_{\lambda(\mathbf{a})=l} \overline{X^{(n)}_{(\mathbf{a})}} = \coprod_{\lambda(\mathbf{b})\leqslant l} X^{(n)}_{(\mathbf{b})}.$$

— $K_l = \coprod_{\lambda(\mathbf{a})=l} K_{\mathbf{a}} \to X^{(n)}_l$.

## 2. THE DOUADY SPACE OF A BI-DISK

In this section we recall the definition and main properties of the Douady space $X^{[n]}$ of "0-dimensional subspaces of length $n$" of a complex surface $X$; see Subsection 2.1. We give a self-contained construction, i.e., without assuming the main existence result of Douady [9], of the Douady space and of the associated Douady–Barlet morphism in the case of $\mathbb{C}^2$ and of the bi-disk $\Delta$; see Subsection 2.2. We do the same thing for *any* complex surface using a patching argument; see Subsection 2.3.

### 2.1. *The Douady Spaces $D(X)$ and $X^{[n]}$*

Let *An* be the category of analytic spaces and $X$ be an analytic space. A *family of compact subspaces of $X$ parameterized by an analytic space $S$* is, by definition, an analytic subspace of $S \times X$ proper and flat over $S$. Let $\Phi$ be the contravariant functor which assigns to any analytic space $T$ the set of families of compact subspaces of $X$ parameterized by $T$. A fundamental result of A. Douady [9] asserts that the functor $\Phi$ is representable, i.e., there exists, unique up to unique isomorphism, a complex space $D(X)$ such that $\Phi \simeq Mor_{An}(-, D(X))$ as functors; the analytic space $D(X)$ is called the *Douady space of X*. By definition, there is a universal flat family $u\colon \mathcal{U}_X \to D(X)$.

Let $X$ be a quasi-projective complex scheme. Let $H(X)$ be the associated Hilbert scheme and $H(X)_{An}$ be the associated analytic space. The universal family over $H(X)$ gives a holomorphic family over $H(X)_{An}$ and therefore, by virtue of the universal property and of Chow Theorem, there is a natural bijective morphism $f\colon H(X)_{An} \to D(X)$. First order algebraic deformations induce analytic ones so that the differential $df$ is injective at every point. Let $t$ be a point in $H(X)_{An}$ and $f(t)$ be the corresponding point in $D(X)$. By "GAGA" principles the corresponding Zariski tangent spaces have the same dimension. It follows that $df$ is also surjective at every point, i.e., $df$ is an isomorphism at every point. In particular, $f\colon H(\mathbb{A}^2_{\mathbb{C}})_{An} \simeq D(\mathbb{C}^2)$: in fact these spaces are both smooth (see Theorem 2.2.1). The same is true for every smooth algebraic surface, as long as we consider those components parameterizing zero-dimensional components. We thank R. Vakil for a useful conversation concerning this point.

*Remark* 2.1.1. If $X$ is projective, then every connected component of $H(X)$ is projective and, in particular, it is compact. This is no longer true for an arbitrary (e.g., non-Kähler) compact complex manifold; see [31]. But things can get even worse. Consider Douady's example of the complex singular space $X$ constructed by identifying a line and a conic in $\mathbb{P}^3$. One can construct a connected component $D$ of $D(X)$ with an infinite number of irreducible components $D_i$; each $D_i$ is compact, but $\sup_i\{\dim D_i\} = +\infty$ and the number of connected components of the members of the family over $D$ can be made to be arbitrarily large. This example grew out of conversations with T. Graber, A. J. de Jong and R. Vakil. The usual example, due to Hironaka, of a compact smooth threefold which is not a scheme leads to pathologies for the Barlet space of cycles, but, apparently, not for the Douady space.

The kind of pathologies described above do not occur for the Douady space of zero-dimensional subspaces of a complex surface (and more generally of a smooth $n$-fold). Let $X$ be an analytic space and consider zero-dimensional analytic subspaces: the flatness of $u\colon \mathcal{U}_X \to D(X)$ decomposes the open and closed subset of $D(X)$ corresponding to zero-dimensional families into the disjoint union of the connected components over which the family has a fixed degree. Let $\Phi^n$ be the sub-functor of $\Phi$ corresponding to those families which are flat and finite of degree $n$ over the base. By what above, the functor $\Phi^n$ is represented by an open and closed subspace $X^{[n]} \subseteq D(X)$. If $X$ is a compact complex surface, then $X^{[n]}$ is compact; see Theorem 2.3.1.

Let $j: U \to X$ be an open immersion of complex spaces. Then $D(U)$ sits naturally (with respect to $j$) inside of $D(X)$. In particular, $\Delta^{[n]} \subseteq \mathbb{C}^{2[n]} = (\mathbb{A}^2_{\mathbb{C}})^{[n]}_{An}$. This open immersion is made explicit below via the "toy model" which we describe momentarily.

### 2.2. *The Toy Model for* $\mathbb{C}^{2[n]}$ *and* $\Delta^{[n]}$

An explicit construction of the Hilbert schemes of $n$-points $(\mathbb{A}^2_{\mathbb{C}})^{[n]}$, based on its existence, which was proved by A. Grothendieck, can be found in [28, Sect. 1].

We now show how the construction satisfies the universal property so that we provide a self-contained and complete construction of $(\mathbb{A}^2_{\mathbb{C}})^{[n]}$ which is independent of the usual general existence result. The proof works algebraically as well as analytically and gives the existence of the Douady spaces $\mathbb{C}^{2[n]}$ which is independent of [9]. This construction also establishes the existence of the Douady spaces $\Delta^{[n]}$ and it identifies them concretely as sitting inside of $\mathbb{C}^{2[n]}$.

The construction of [28, Sect. 1] proves that

$$\mathscr{P}^n := \{(A, B, v) \in \mathbf{gl}(n) \times \mathbf{gl}(n) \times \mathbb{C}^n \mid [A, B] = 0, \mathbb{C}^n = Span\{A^k B^l v\}_{k, l \in \mathbb{N}}\}$$

is connected and smooth, that it carries a flat family $w: \mathscr{W}^n \to \mathscr{P}^n$ of degree $n$, that $\mathbf{GL}(n)$ acts freely on $\mathscr{P}^n$ by $G(A, B, v) = (GAG^{-1}, GBG^{-1}, Gv)$ and that the quotient $\mathfrak{g}: \mathscr{P}^n \to \mathscr{Q}^n := \mathscr{P}^n / \mathbf{GL}(n)$: (1) exists, (2) is connected and smooth and (3) that the family on $\mathscr{P}^n$ goes to the quotient and defines a flat family $v: \mathscr{V}^n \to \mathscr{Q}^n$ of degree $n$. We now show that the family over $\mathscr{Q}^n$ is universal. We prove the result in $An$.

THEOREM 2.2.1. *Let $X = \mathbb{C}^2$. The functor $\Phi^n$ is represented by $(\mathscr{Q}^n_{An}, v)$, i.e., $\mathbb{C}^{2[n]}$ exists and is isomorphic to $\mathscr{Q}^n$.*

*Proof.* Since the functor $\Phi^n$ is a sheaf of sets with respect to the classical topology, it is enough to show that $\mathscr{Q}^n$ enjoys the universal property with respect to *germs of families* $\varphi: F \to S$ parameterized by analytic germs $(S, s)$, i.e., we need to show that if $F \subseteq S \times \mathbb{C}^2$ is finite and flat of degree $n$ with respect to the first projection, then there exists a unique morphism of germs $\alpha(\varphi): (S, s) \to (\mathscr{Q}^n, q)$ such that the germ of families $\alpha(\varphi)^* \mathscr{V}^n$ *is* $\varphi: F \to S$.

Let $\varphi: F \to S$ be such a germ. The coherent $\mathscr{O}_S$-module $M := \varphi_* \mathscr{O}_F$ is free of rank $n$. The coordinates $(z_1, z_2)$, acting by multiplication on $M$, define two commuting $\mathscr{O}_S$-linear endomorphism $T_1$ and $T_2$ of $M$. The distinguished section $1 \in M$ generates $M$ under the action of the monomials $T_1^k T_2^l$.

Choose a $\mathscr{O}_S$-linear isomorphism $\varepsilon: M \to \mathscr{O}_S^{\oplus n}$. By specializing at $s$, and using the trivialization $\varepsilon$, we get $p := (A, B, v) \in \mathscr{P}^n$. By virtue of [13, 0.21], this datum is equivalent to giving a morphism of germs $\tilde{\alpha}(\varphi)_\varepsilon: (S, s) \to (\mathscr{P}^n, p)$. By construction $\tilde{\alpha}(\varphi)_\varepsilon^* \mathscr{W}^n$ *is* $\varphi: F \to S$.

By composition we get $\alpha(\varphi) := \mathfrak{g} \circ \tilde{\alpha}(\varphi)_\varepsilon: (S, s) \to (\mathscr{Q}^n, q := \mathfrak{g}(p))$ with the property that $\alpha(\varphi)^* \mathscr{V}^n$ *is* $\varphi: F \to S$. The morphism $\alpha(\varphi)_\varepsilon$ is independent of $\varepsilon$ so that we denote it by $\alpha(\varphi)$. To prove uniqueness, note that if $\eta: (S, s') \to (\mathscr{Q}^n, q')$ is any morphism of germs, then we get a germ of families $\sigma: \eta^* \mathscr{V}^n \to S$, and $\alpha(\sigma) = \eta$. ∎

We now recall the definition of the Douady–Barlet map. The starting point is a "doubled" version of the map $\mathbf{gl}(n) \to \mathbf{gl}(n)/Ad \simeq \mathbb{C}^{(n)}$ which associates with a matrix its characteristic polynomial. By a classical theorem in Invariant Theory (cf. [34]), the ring of invariant functions of $2n$ variables, $((x_1, y_1), ..., (x_n, y_n))$, by the action of the symmetric group $\mathfrak{S}_n$ is generated by the functions $f_{k,l}((x_1, y_1), ..., (x_n, y_n)) = \sum_i x_i^k y_i^l$. In other words, $\mathbb{C}^{2^{(n)}} = Spec_{An}\, \mathbb{C}[x_1, y_1, ..., x_n, y_n]^{\mathfrak{S}_n} = Spec_{An}\, \mathbb{C}[..., f_{k,l}, ...]$.

Let $(A, B, v) \in \mathscr{P}^n$. Since $A$ and $B$ commute, we can let the group $\mathbf{GL}(n)$ act so that they are both in triangular form. If $x_1, ..., x_n$ and $y_1, ..., y_n$ are the respective diagonal terms, well defined up to a permutation, then $f_{k,l}(x_1, y_1, ..., x_n, y_n) = \sum_i x_i^k y_i^l = Trace(A^k B^l)$. These functions are $\mathbf{GL}(n)$-invariant. We can therefore define a morphism $\pi: \mathbb{C}^{2^{[n]}} \to \mathbb{C}^{2^{(n)}}$, which is called the *Douady–Barlet* morphism, or the Hilbert–Chow morphism in the algebraic category. In set-theoretic terms this map associates with a subspace of length $n$ of $\mathbb{C}^2$ its support, counting multiplicities, seen as a point in $\mathbb{C}^{2^{(n)}}$. The Douady–Barlet morphism is easily seen to be proper and its fibers are connected by Zariski Main Theorem.

Let $U$ be an open subset of $\mathbb{C}^2$. Since the quotient map $(\mathbb{C}^2)^n \to \mathbb{C}^{2^{(n)}}$ is finite and therefore open, $U^{(n)}$ is open in $\mathbb{C}^{2^{(n)}}$ and $U^{[n]}$ ($\pi: U^{[n]} \to U^{(n)}$, respectively) can be naturally identified with $\pi^{-1}(U^{(n)})$ ($\pi: \pi^{-1}(U^{(n)}) \to U^{(n)}$, respectively). In particular, we have

PROPOSITION 2.2.2. *Let* $\Delta = \{(z_1, z_2) \in \mathbb{C}^2 : |z_i| < 1, i = 1, 2\}$ *be the two-dimensional bi-disk. The Douady space* $\Delta^{[n]}$ *can be described as the quotient by the action of* $\mathbf{GL}(n)$ *of*

$$\mathscr{P}^n_\Delta := \{(A, B, v) \in \mathscr{U}^n \mid \textit{the eigenvalues of } A \textit{ and } B \textit{ have modulus smaller than one}\}.$$

*Remark* 2.2.3. The natural action of $\mathbb{C}^{*2}$ on $\mathbb{C}^2$ induces an action on $\mathbb{C}^{2^{[n]}}$ given by $(\lambda_1, \lambda_2)(A, B, v) = (\lambda_1 A, \lambda_2 B, v)$. The subspace $\Delta^{[n]}$ is $(S^1)^2$-invariant under this action. The fixed points are easily determined and they correspond to the partitions of $n$; see [10, Sect. 3; 28, Sect. 5].

The following is a nice consequence of the toy model developed above.

PROPOSITION 2.2.4. *The manifolds* $\mathbb{C}^{2^{[n]}}$ *and* $\Delta^{[n]}$ *are homeomorphic to each other. In particular,* $\mathbb{C}^{2^{[n]}}$ *and* $\Delta^{[n]}$ *have the same Betti numbers.*

*Proof.* Let $\phi(A, B) :=$ maximum of the absolute values of the eigenvalues of $A$ and $B$. Consider the application $\Phi: \Delta^{[n]} \to \mathbb{C}^{2^{[n]}}$ defined as

$$\Phi(A, B, v) = \left( \frac{1}{1-\phi(A, B)} A, \frac{1}{1-\phi(A, B)} B, v \right).$$

This map is $\mathbf{GL}(n)$-invariant and defines a homeomorphism. ∎

At this point we know the Betti numbers of $\Delta^{[n]}$ by virtue of Proposition 2.2.4 and of [10, Theorem 1.1] (since loc. cit. has a typo, the reader is referred to [28, 5.9]). However, our analysis of $\pi: X^{[n]} \to X^{(n)}$ will require a certain basis for the cohomology which reflects the geometry of the stratified Douady–Barlet morphism. We shall obtain such a basis via Nakajima's construction. Compare with Remark 2.5.2.

## 2.3. *Construction of* $X^{[n]}$ *and* $\pi: X^{[n]} \to X^{(n)}$

We can now sketch a proof of the existence of $X^{[n]}$ and of $\pi: X^{[n]} \to X^{(n)}$ by a patching argument. In the algebraic case (and assuming the existence of the Hilbert scheme) the corresponding assertions are due to Fogarty [14].

THEOREM 2.3.1. *Let* $X$ *be a complex surface. For every* $n \in \mathbb{N}$, *the Douady space* $X^{[n]}$ *and the Douady–Barlet morphism* $\pi: X^{[n]} \to X^{(n)}$ *exist and can be constructed by patching using* $\Delta^{[m]}$ *and* $\pi: \Delta^{[m]} \to \Delta^{(m)}$, $m \leqslant n$. *The space* $X^{[n]}$ *is a connected 2n-dimensional complex manifold. The Douady–Barlet morphism is projective. In particular, if* $X$ *is compact, then* $X^{[n]}$ *is compact.*

*Sketch of Proof.* We freely use the local descriptions given in Section 3. Let $z \in X^{(n)}$ be any point. There is a unique partition $\tilde{\nu} \in P(n)$ such that $z = \sum_{j=1}^{k=\lambda(\tilde{\nu})} \nu_j x_j \in X^{(n)}_{(\tilde{\nu})}$, where the points $x_j \in X$ are pairwise distinct. Consider the complex manifold $\prod_{j=1}^{k} {}_{x_j}\Delta^{[\nu_j]}$. This way, we obtain a set of charts which glue coherently by the universal property of the Douady space (which we have already constructed for $\Delta$) and define a connected complex manifold $W^n$. Similarly, one can check that the local Douady–Barlet morphisms glue and define a global proper map with connected fibers $\pi: W^n \to X^{(n)}$. By construction $W^n$ carries a family. It is elementary to check the universal property of this family using the corresponding fact for $\Delta$ and by the fact that the functor $\Phi$ is a sheaf for the classical topology.

The Douady–Barlet morphism is bimeromorphic. It is also projective, locally over $X^{(n)}$. We conclude by the fact that there is a unique irreducible exceptional divisor, i.e., the pre-image of the big diagonal in $X^{(n)}$. ∎

*Remark* 2.3.2. An alternative definition of $\pi$ was given by Iversen [24]. More generally, D. Barlet has constructed the so-called Barlet spaces of cycles $B(X)$ associated with any analytic space $X$ and a natural morphism $D(X)_{red} \to B(X)$ which, in our case, gives $\pi\colon X^{[n]} \to X^{(n)}$. This explains the name Douady–Barlet for the morphisms $\pi$. One can show that this morphism is proper if we consider zero-dimensional subspaces of length $n$ on a smooth manifold.

## 2.4. *The Operators of Nakajima*

Let $X$ be a complex surface. In what follows we work with rational singular cohomology. We now recall, for the convenience of the reader, Nakajima's construction [27, 28] of the correspondences which realize geometrically an action of the Heisenberg/Clifford super-algebra on

$$\mathbb{H}(X) := \bigoplus_{r \geqslant 0} H^*(X^{[r]}).$$

Let $Y$ be a topological manifold of dimension $m$; we denote rational cohomology simply by $H^*(Y)$ and we identify it freely with Borel–Moore homology $H^{lf}_{m-*}(Y)$ via Poincaré Duality.

For every $r \geqslant 0$ and every $k > 0$, define a closed and reduced analytic subspace $T_k \subseteq X \times X^{[r]} \times X^{[r+k]}$ by setting

$$T_k = \{(x, \zeta_1, \zeta_2) : \mathcal{I}_{\zeta_2} \subseteq \mathcal{I}_{\zeta_1},\ \mathcal{I}_{\zeta_1}/\mathcal{I}_{\zeta_2} \text{ is supported at } x\},$$

where $\mathcal{I}_{\zeta_i}$, $i = 1, 2$, are the coherent sheaves of ideals associated with the closed analytic space $\zeta_i \subseteq X$, $i = 1, 2$.

By [28, Sect. 8.3], $T_k$ has a unique irreducible reduced component $Z_k$ of (expected) dimension $2 + 2r + (k-1)$ and all other irreducible components, if there are any at all, have strictly lower dimension. We ignore whether $T_k$ is irreducible (i.e., $T_k = Z_k$) or not. However, this does not cause any problem in what follows.

We are implicitly making use of Briançon's irreducibility result [6, V.3.3] as in [27, 28]. However, this turns out the be not necessary; see Subsection 2.6. At this stage, we prefer to use this result for clarity of exposition.

The projections $p_{12}\colon X \times X^{[r]} \times X^{[r+k]} \to X \times X^{[r]}$ and $p_3\colon X \times X^{[r]} \times X^{[r+k]} \to X^{[r+k]}$ are proper when restricted to $Z_k$. Therefore we can define the two correspondences, one transpose of the other,

$$\mathscr{P}[k] := p_{3*}(p_{12}^{*}(\cdots) \cap Z_k)\colon H^*(X \times X^{[r]}) \to H^*(X^{[r+k]}),$$
$$\mathscr{R}[k] := p_{12*}(p_3^{*}(\cdots) \cap Z_k)\colon H^*(X^{[r+k]}) \to H^*(X \times X^{[r]}),$$

or, equivalently,

$$\tilde{\mathscr{P}}[k]\colon H^*(X) \to Hom(H^*(X^{[r]}), H^*(X^{[r+k]})),$$
$$\tilde{\mathscr{R}}[k] = H_c^*(X) \to Hom(H^*(X^{[r+k]}, H^*(X^{[r]})).$$

Putting all $r$'s together we obtain endomorphisms of $\mathbb{H}(X)$. The endomorphisms corresponding to $\alpha \in H^*(X)$ and $\beta \in H_c^*(X)$ will be denoted by $\tilde{\mathscr{P}}_\alpha[k]$ and $\tilde{\mathscr{R}}_\beta[k]$, respectively.

Since $\dim Z_k = k+1+2r$ and $\dim X \times X^{[r]} \times X^{[r+k]} = 2(k+1+2r)$, it is easy to compute that the operators $\mathscr{P}[k]$ increase the degree by $2(k-1)$ and that the operators $\mathscr{R}[k]$ decrease the degree by the same amount, and similarly if we look at the correspondences on homology groups or on locally finite homology groups.

The following is an explicit, simple, but important exemplification of how these operators work.

LEMMA 2.4.1. *Let $X$ be any complex surface, $\tilde{v} = (v_1, ..., v_k) = \mathbf{a} \in P(n)$, $\mathbf{a}! := \prod_{i=1}^n a_i!$, $[\overline{X_{(\tilde{v})}^{[n]}}] \in H^{2(n-k)}(X^{[n]}, \mathbb{Q})$ be the fundamental class of $\overline{X_{(\tilde{v})}^{[n]}}$. Then*

$$\tilde{\mathscr{P}}_{[X]}[v_1] \circ \cdots \circ \tilde{\mathscr{P}}_{[X]}[v_k](\mathbf{1}) = \mathbf{a}!\, [\overline{X_{(\tilde{v})}^{[n]}}].$$

*Proof.* By induction on the length $k$ of $\tilde{v}$, it is enough to prove that

$$\tilde{\mathscr{P}}_{[X]}[v_k]\,([\overline{X_{(v_1, ..., v_{k-1})}^{[n-v_k]}}]) = a_{v_k}\, [\overline{X_{(\tilde{v})}^{[n]}}].$$

To compute the left hand-side we first consider the intersection $T := p_1^{-1}(X) \cap p_2^{-1}(\overline{X_{(v_1, ..., v_{k-1})}^{[n-v_k]}}) \cap Z_k$ inside of $X \times X^{[n-v_k]} \times X^{[n]}$. The space $T$ is irreducible of the expected dimension and the intersection is transverse at a general point of $T$. The third projection $p_{3_|}\colon T \to X^{[n]}$ is a proper, generically finite morphism onto its image $\overline{X_{(\tilde{v})}^{[n]}}$. We can determine the generic degree at a general point. This degree is $a_{v_k}$. ∎

*Remark* 2.4.2. Let $X$ be a Zariski-dense open subset of a compact Kähler surface $Y$, e.g. $X$ is quasi-projective. Then $X^{[n]}$ is a Zariski-dense open subset of the compact Kähler manifold $Y^{[n]}$. It follows that $H^*(X)$ and $H^*(X^{[r]})$ have a Hodge decomposition. Since the class $Z_k$ has type $(k+2r+1, k+2r+1)$ and $p_{3*}$ has degree $(-(2+2r), -(2+2r))$, it follows that $\mathscr{P}[k]$ and $\mathscr{R}[k]$, have degree $(k-1, k-1)$ and $(1-k, 1-k)$, respectively.

If $\alpha$ has type $(p, q)$, then $\widetilde{\mathscr{P}}_\alpha[k]\colon H^{*,*}(X^{[r]}) \to H^{*+p+k-1,\, *+q+k-1}(X^{[r+k]})$.

As anticipated, Nakajima's construction and Göttsche's formula can be conveniently encoded in representation-theoretic terms. For this it is necessary to introduce the language of super ($=\mathbb{Z}_2$-graded) algebras. A general reference is [25].

Let us consider $H^*(X) = H^{\mathrm{even}} \oplus H^{\mathrm{odd}}$ as a super vector space, and define the Heisenberg superalgebra

$$\mathscr{H}(X) := \mathbb{Q}\langle c\rangle \oplus \mathbf{s}_{>0} \oplus \mathbf{s}_{<0},$$

where

$$\mathbf{s}_{>0} = \bigoplus_{i>0} H^*(X)[i], \qquad \mathbf{s}_{<0} = \bigoplus_{i<0} H_c^*(X)[i]$$

(here $[i]$ is just a place holder and does not refer to any shift) the $\mathbb{Z}_2$-grading comes from the one on $H^*(X)$, the $\mathbb{Z}$-degree of $c$ is zero, the $\mathbb{Z}_2$-degree of $c$ is even, and the defining properties are

(a) $c$ is central,

(b) $\mathbf{s}_{>0}$ and $\mathbf{s}_{<0}$ are supercommutative, i.e., the supercommutator is identically zero,

(c) $[\beta[j], \alpha[i]] = \delta_{i,-j}(-1)^{i-1}\, i\langle \alpha, \beta\rangle\, c$,

where $\langle -, -\rangle$ is the canonical Poincaré pairing between cohomology and cohomology with compact supports. Note that the factors $(-1)^{i-1} i$ are not necessary and could be either omitted, or replaced by another compatible system of factors. We have used them in view of Theorem 2.4.3 where they appear naturally in connection with the computation of a certain intersection number which has been determined in [12].

Since $\mathbf{s}_{>0}$ is supercommutative, the enveloping superalgebra $U(\mathbf{s}_{>0})$ (which is constructed like its non-super analogue by factoring the tensor algebra by the ideal $x \otimes y - (-1)^{\deg x \deg y} y \otimes x - [x, y]$, where $[-, -]$ is the supercommutator) is the free commutative superalgebra on $H^*(X)$ isomorphic to $S^*(\mathbf{s}_{>0}^{\mathrm{even}}) \otimes \bigwedge^* (\mathbf{s}_{>0}^{\mathrm{odd}})$. By its own definition, $\mathbf{s}_{>0}$ is $\mathbb{N}^+$-graded, and an $\mathbb{N}$-grading is inherited by $U(\mathbf{s}_{>0})$.

If $\dim H^*(X) < \infty$, then a computation shows that

$$\sum_{n=0}^{\infty} q^n \dim U(\mathbf{s}_{>0})_n = \prod_{m=1}^{\infty} \frac{(1+q^m)^{\dim H^{\mathrm{odd}}(X)}}{(1-q^m)^{\dim H^{\mathrm{even}}(X)}}. \tag{1}$$

Denote by $[-, -]$ the supercommutator in $\operatorname{End} \mathbb{H}(X)$. The main theorem in [27; 28, Sect. 8] is

THEOREM 2.4.3. *The following commutation relations hold*:

$$[\tilde{\mathscr{P}}_\alpha[k], \tilde{\mathscr{P}}_\gamma[l]] = 0,$$
$$[\tilde{\mathscr{R}}_\beta[k], \tilde{\mathscr{R}}_\delta[l]] = 0,$$
$$[\tilde{\mathscr{R}}_\beta[l], \tilde{\mathscr{P}}_\alpha[k]] = \delta_{k,l}(-1)^{k-1}\, k\langle \alpha, \beta\rangle\, Id_{\mathbb{H}(X)}.$$

Note that the "reversed" order in the third relation is due to the fact that we are using operators which are dual to the ones that Nakajima defines in homology (cf. [28, Sect. 8]). Equivalently, the assignments:

$$\alpha[i] \to \tilde{\mathscr{P}}_\alpha[i] \text{ if } i > 0 \text{ and } \alpha \in H^*(X),$$
$$\beta[i] \to \tilde{\mathscr{R}}_\beta[-i] \text{ if } i < 0 \text{ and } \beta \in H^*_c(X),$$
$$c \to Id_{\mathbb{H}(X)}$$

exhibit $\mathbb{H}(X)$ as a representation of $\mathscr{H}(X)$. For every $i > 0$, the vector $\mathbf{1} \in H^0(X^{[0]})$ is annihilated by $\tilde{\mathscr{R}}_\beta[i]$ and is therefore a highest weight vector. Furthermore, it is immediate that, $\forall v \in \mathbb{H}(X)$, $\tilde{\mathscr{R}}_\beta[i]\, v = 0$ for $i$ big enough depending on $v$.

Denote by $U'$ the $\mathscr{H}(X)$-submodule of $\mathbb{H}(X)$ generated by $\mathbf{1}$. It is well known that $U'$ is irreducible and isomorphic to $U(\mathbf{s}_{>0})$. Because of formula (3.1), the Göttsche's formula (see Theorem 5.2.1(2)) becomes the statement $U' = \mathbb{H}(X)$.

Let us summarize Nakajima's construction and its link to the Göttsche Formula.

THEOREM 2.4.4. *Let $X$ be a complex surface. Then*

(2.4.4.1) *$\mathbb{H}(X)$ is a highest weight representation of $\mathscr{H}(X)$, geometrically realized by the operators $\tilde{\mathscr{P}}_\alpha[k]$ and $\tilde{\mathscr{R}}_\beta[l]$.*

(2.4.4.2) *The representation is irreducible iff the Göttsche formula holds for $X$.*

### 2.5. *Irreducibility of the Action for $X = \Delta$ and a Canonical Basis for $H^*(\Delta^{[n]}, \mathbb{Q})$*

We now examine in detail the picture in homology in the case of $\mathbb{C}^2$ and $\Delta$. They can be dealt with simultaneously. Let $X = \mathbb{C}^2$ or $X = \Delta$.

The Heisenberg algebra $\mathscr{H}(X)$ is in this case the standard one and the fundamental representation $U(\mathbf{s}_{>0})$ is isomorphic to the space of polynomials in infinitely many variables $p_i$ to which we assign degree $i$. See [28, Sect. 8]. It follows therefore that, for every $n$, $\dim U(\mathbf{s}_{>0})_n = p(n)$, the number of partitions of $n$.

We can now give a quick and self-contained proof of the formula for the Betti numbers of the Douady space of $\mathbb{C}^2$ or $\Delta$, very much in the spirit of this paper. This formula, and especially the construction of the explicit basis derived from it, are heavily used in the sequel of this paper. For the original proof of the formula for $X = \mathbb{C}^2$, see [10] or the slightly different method in [28].

THEOREM 2.5.1. *Let $X = \mathbb{C}^2$ or $X = \Delta$. Then*

(2.5.1.1) *$\mathbb{H}(X)$ is an irreducible highest weight vector representation of the Heisenberg algebra with highest weight vector the generator $\mathbf{1} \in H^0(X^{[0]}) \simeq \mathbb{Q}$.*

(2.5.1.2) *The Poincaré polynomials of $X^{[n]}$ and their generating function are, respectively*:

$$P_t(X^{[n]}) = \sum_{\tilde{\nu} \in P(n)} t^{2n - 2\lambda(\tilde{\nu})}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} P_t(X^{[n]})\, q^n = \prod_{m=1}^{\infty} \frac{1}{1 - t^{2m-2} q^m}.$$

(2.5.1.3) *The cycles $\overline{X^{[n]}_{(\tilde{\nu})}} \in H^{lf}_{2(n+\lambda(\tilde{\nu}))}(X^{[n]}) = H^{2(n-\lambda(\tilde{\nu}))}(X^{[n]})$ form a basis for $H^*(X^{[n]})$.*

*Proof.* Since the generator of $H^0(X^{[0]})$ is a highest weight vector, it follows that $\mathbb{H}(X)$ contains a subrepresentation isomorphic to $U(\mathbf{s}_{>0})$. In order to prove the irreducibility, it is enough to prove that $\sum_i \dim H^i(X^{[n]}) = \dim U(\mathbf{s}_{>0})_n = p(n)$, for every $n$. As we already noted, $(\mathbb{C}^2)^{[n]}$ has an action of $(\mathbb{C}^*)^2$ with isolated fixed points. It is possible to choose a one-parameter subgroup $\mathbb{C}^* \subseteq (\mathbb{C}^*)^2$ acting with the same fixed point set and such that $\lim_{t \to 0} t \cdot x$ exists, for every $x \in (\mathbb{C}^2)^{[n]}$. This is enough to obtain a Bialynicky–Birula-type cell decomposition of $(\mathbb{C}^2)^{[n]}$. It follows that the odd Betti numbers vanish and that the sum of the even Betti numbers is equal to the number of these fixed points. In this case they correspond to monomial ideals of colength $n$ and the number of these is exactly $p(n)$. This proves the first part also for $\Delta$ by virtue of Proposition 2.2.4.

Taking into account the degree properties of the operators $\tilde{\mathscr{P}}_{[X]}[l]$, the second statement follows immediately from the first one.

The third one follows from Lemma 2.4.1. ∎

*Remark* 2.5.2. Theorem 2.5.1 gives a different proof of [10, Theorem 1.1(iii)] and determines a basis of elements for $\mathbb{H}(\mathbb{C}^2)$ different from the one in [11]. The basis (2.5.1.3) for $\mathbb{H}(\Delta)$ is used in an essential way in our study of the Douady–Barlet morphism.

*Remark* 2.5.3. Let $X$ be either $\mathbb{C}^2$, or $\Delta$. The algebra $\mathscr{H}(X)$ is the standard Heisenberg algebra. The Göttsche Formula holds for $X$. See [10] and Proposition 2.2.4; see also Theorem 2.5.1. It follows that $\mathbb{H}(X) = \langle \mathbf{1} \rangle_{\mathscr{H}(X)}$. Recall that $\langle \mathbf{1} \rangle_{\mathscr{H}(X)} \simeq \mathbb{Q}[t_1, t_2, ...]$ as $\mathscr{H}(X)$-modules; see [28, Sect. 8], for example, and assign $t_i$ to $\tilde{\mathscr{P}}_{[X]}[i](\mathbf{1})$. Let $\tilde{\nu} \in P(n)$. By virtue of Lemma 2.4.1 and after an obvious normalization, we see that the monomial $t_{\nu_1} \cdots t_{\nu_k}$ corresponds to the class of $\overline{X^{[n]}_{(\tilde{\nu})}}$. This fits nicely with Theorem 2.5.1.3.

## 2.6. *Punctual Hilbert Schemes: Betti Numbers and Irreducibility*

Let $n \in \mathbb{N}$, $p \in X$ and $X^{[n]}_p$ be the fiber of $\pi$ over the point $np \in X^{(n)}_{(n)}$ with the induced reduced structure. These spaces are analytically isomorphic to the corresponding spaces $\Delta^{[n]}_o$ when $X$ is $\Delta$ (or $\mathbb{C}^2$) and $p$ is the origin $o$. In particular they are projective. They are called *punctual Hilbert schemes.*

We can give a new proof, based on Theorem 2.5.1 of the following basic results [10, Theorem 3.1(iv); 6, V.3.3] concerning some of the topology of these spaces.

COROLLARY 2.6.1 (cf. [10]). *The Poincaré polynomials of $\Delta^{[n]}_o$ and their generating function are*

$$P_t(\Delta^{[n]}_o) = \sum_{\tilde{\nu} \in P(n)} t^{2n - 2\lambda(\tilde{\nu})}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} P_t(\Delta^{[n]}_o)\, q^n = \prod_{m=1}^{\infty} \frac{1}{1 - t^{2m-2} q^m}.$$

*Proof.* It follows at once from cohomology and base change for proper maps and Theorem 2.5.1.3. ∎

The space $\Delta^{[n]}_o$ is irreducible by a result of Briançon; see [6, V.3.3]; this result is re-proved by different methods in [10, Corollary 1.2].

We want to observe that Briançon's result can be re-proved, in the spirit of this paper, using Theorem 2.5.1 above, exactly as [10, Theorem 1.1.iv] is used to prove [10, Corollary 1.2]. Theorem 2.5.1 is based on the use of Nakajima construction which uses Briançon's result. (It should be pointed out that the hard part of this irreducibility result has to do with bounding from below certain loci; once that has been achieved, the irreducibility follows by standard arguments not involving the indirect argument reproduced below; besides Briançon's original proof, we would like to mention work of Gaffney, Gaffney–Lazarsfeld, Granger and Iarrobino; see [23].) To avoid biting our tail we observe that a modification in the definition of the varieties $Z_k$ in Subsection 2.4 allows us to modify slightly Nakajima's construction without using the irreducibility of punctual Hilbert schemes.

Let $CX^{[n]}_p$ be the closure in $X^{[n]}_p$ of the locus of curvilinear subspaces supported at $p$. It is irreducible of dimension $n-1$ (cf. the end of Section in [28]). Note that the irreducibility of $\Delta^{[n]}_o$ is equivalent to $CX^{[n]}_p = X^{[n]}_p$.

Define

$$S_k := \left\{ (x, \zeta_1, \zeta_2) \in X \times X^{[n]} \times X^{[n+k]} \,\middle|\, x \notin Supp(\zeta_1),\ \zeta_2 = \zeta_1 \coprod \zeta_2',\ \zeta_2' \in CX_p^{[k]} \right\},$$

and $Z_k' = \overline{S_k}$. Nakajima's construction works if we replace $Z_k$ by $Z_k'$. We thus have

Corollary 2.6.2 (cf. [6, 10]). *The spaces $\Delta_o^{[n]}$ are irreducible.*

*Proof.* By a result of Gaffney and Lazarsfeld [22, Theorem 2], all the irreducible components of $\Delta_o^{[n]}$ have dimension at least $n-1$. By virtue of Theorem 2.5.1, we have that (i) $b_{2(n-1)}(\Delta_o^{[n]}) = 1$ and (ii) $b_t(\Delta_o^{[n]}) = 0$, for every $t > 2(n-1)$. The irreducibility follows easily. ∎

## 3. STUDY OF THE LOCAL STRUCTURE OF THE DOUADY–BARLET MORPHISM

This section consists of the detailed analysis of the stratified morphism $\pi: X^{[n]} \to X^{(n)}$. The goal is to define the morphism of complexes $\Psi$; see Proposition 3.6.2. The morphism $\Psi$ is defined via its components $\Psi^{2h}$ (the odd ones are zero). The morphisms of sheaves $\Psi^{2h}$ are defined by establishing natural identifications between combinatorial and geometric objects arising from our analysis of the stratified morphism. In short, we first fix a point $z \in X^{(n)}$ and small natural euclidean neighborhoods $\mathfrak{U}$ of $z$, then we identify the fibers over $z$ of *all* the normalization morphisms $K_{\mathbf{a}}$, for those $\mathbf{a}$ such that $\lambda(\mathbf{a}) = n-h$, with the closed currents of integration associated with (2.5.1.3), which give a basis for the cohomology in degree $2h$ of the open sets $\pi^{-1}(\mathfrak{U})$.

The Decomposition Theorem 4.1.1 will be the statement that $\Psi$ is a quasi-isomorphism.

### 3.1. *Partial Ordering on $P(n)$*

Let $\tilde{\nu} := (\nu_1, ..., \nu_k)$, $\tilde{\nu}' := (\nu_1', ..., \nu_l')$ be two partitions of $n$. We say that $\tilde{\nu}' \geqslant \tilde{\nu}$ if there exists a partition of the set $\{\nu_1', ..., \nu_k'\}$ into $l$ disjoint subsets $\Gamma_j := \{\mu_1^j, ..., \mu_{t_j}^j\}$ such that the elements of $\Gamma_j$ form a partition of $\nu_j$, for every $j = 1, ..., k$. Geometrically, $\tilde{\nu}' \geqslant \tilde{\nu}$ iff $X_{(\tilde{\nu})}^{(n)} \subseteq \overline{X_{(\tilde{\nu}')}^{(n)}}$.

Example 3.1.1. Let $n = 6$; one sees immediately that the pairs (3, 1, 1, 1) and (2, 2, 2), (4, 1, 1) and (3, 3), (2, 2, 2) and (5, 1), (2, 2, 2) and

(3, 3) are not comparable in the sense that neither element of the pair is greater than or equal to the remaining element of the pair.

It may be useful to arrange all partitions of $n$, in $\tilde{\nu}$-notation, in columns left to right following the decreasing length and draw either solid arrows or dotted segments from the elements of one column to the ones of the column immediately to the right according to whether or not two partitions are comparable. One finds that $\tilde{\nu} \geqslant \tilde{\mu}$ iff there is a path given by solid arrows from left to right connecting $\tilde{\nu}$ and $\tilde{\mu}$.

### 3.2. *A Fundamental System of Neighborhoods on $X^{(n)}$*

Recall that, given a point $y$ in a topological space $Y$ and an indexing set $I$, a *fundamental system of neighborhoods of $y$ in $Y$ labeled by $I$* is a collection $U_i$, $i \in I$ of open neighborhoods of $y$, such that given any open set in $Y$ containing $y$, there exists an index $j \in I$ such that $U_j \subseteq U$. Given such a system for every point $y$ in $Y$, the collection of these open sets forms a basis for the topology of $Y$.

Let $\tilde{\nu} \in P(n)$ and $z = \sum_{j=1}^{k} \nu_j x_j \in X^{(n)}_{(\tilde{\nu})}$, i.e., the points $x_j \in X$ are pairwise distinct. Note that any point $z \in X^{(n)}$ is of this form for a unique partition of $n$. If necessary, we denote the partition associated with $z$ by $\tilde{\nu}(z)$.
Let ${}_{x_j}\Delta$ be a collection of open neighborhoods of the points $x_j \in X$ subject to the following two conditions:

(1) they are pairwise disjoint;

(2) each ${}_{x_j}\Delta$ is biholomorphic to a unit bi-disk $\Delta \subseteq \mathbb{C}^2$ by a fixed isomorphism $f_{x_j}\colon \Delta \to {}_{x_j}\Delta$.

Let $0 < \varepsilon \leqslant 1$ be a real number. Consider $\Delta(\varepsilon) := \{(z_1, z_2) \in \Delta : |z_i| < \varepsilon, i = 1, 2\}$ and define ${}_{x_j}\Delta(\varepsilon) := f_{x_j}(\Delta(\varepsilon))$: it is a neighborhood of $x_j$ in $X$.

By abuse of notation, we denote the $m$-fold symmetric products $({}_{x_j}\Delta(\varepsilon))^{(m)}$ by ${}_{x_j}\Delta^{(m)}(\varepsilon)$. By virtue of the defining property of symmetric products in the category of complex spaces, the cartesian products

$$\prod_{j=1}^{k} {}_{x_j}\Delta^{(\nu_j)}(\varepsilon)$$

are naturally biholomorphic to neighborhoods of $z$ in $X^{(n)}$. We denote them by $U_z(\varepsilon)$.

If $z$ is fixed, then the open sets $U_z(\varepsilon)$, $0 < \varepsilon \leqslant 1$, form a fundamental system of neighborhoods of $z$ in $X^{(n)}$.

If $z$ varies as well, then the open sets $U_z(\varepsilon)$ form a basis for the topology of $X^{(n)}$. We call these neighborhoods *basic*.

Note that: (1) if $z \in X^{(n)}_{(\tilde{\nu})}$, then $U_z(1)$ does not meet any stratum corresponding to a partition $\tilde{\mu} \neq \tilde{\nu}$ such that $\lambda(\tilde{\nu}) \geqslant \lambda(\tilde{\mu})$, and (2) if $U_z(1) \cap U_w(1) \neq \emptyset$, then either $\tilde{\nu}(w) \geqslant \tilde{\nu}(z)$ (i.e., $X^{(n)}_{(\tilde{\nu}(z))} \subseteq \overline{X^{(n)}_{(\tilde{\nu}(w))}}$), or $\tilde{\nu}(z) \geqslant \tilde{\nu}(w)$.

### 3.3. *The Spaces $X^{(\mathbf{a})}$ and the Morphisms $K_{\mathbf{a}}$*

Let $\mathbf{a} \in P(n)$ be a fixed partition of $n$ in $a$-notation. Set $X^{(0)} := pt$, where $pt$ is a single fixed point viewed as a complex space. Define

$$X^{(\mathbf{a})} := \prod_{i=1}^{n} X^{(a_i)}.$$

By abuse of notation, a point in $X^{(\mathbf{a})}$ will be denoted by an $n$-tuple

$$\tau := (x_1^1 + \cdots + x_{a_1}^1, \ldots, x_1^n + \cdots + x_{a_n}^n),$$

where it is understood that for every index $i$ for which $a_i = 0$, the corresponding entry $(x_1^i + \cdots + x_{a_i}^i)$ is to be replaced by the point $pt$.

By the defining property of symmetric products in the category of complex spaces, there is a morphism

$$K_{\mathbf{a}} \colon X^{(\mathbf{a})} \to X^{(n)},$$

$$(x_1^1 + \cdots + x_{a_1}^1, \ldots, x_1^n + \cdots + x_{a_n}^n) \longrightarrow \sum_{i=1}^{n} i(x_1^i + \cdots + x_{a_i}^i),$$

where, by abuse of notation, for every index $i$ for which $a_i = 0$ the corresponding summand in the sum on the right is omitted.

The image of $K_{\mathbf{a}}$ is the closure $\overline{X^{(n)}_{(\mathbf{a})}}$ of the stratum associated with $\mathbf{a}$.

Lemma 3.3.1. *The morphism*

$$K_{\mathbf{a}} \colon X^{(\mathbf{a})} \to \overline{X^{(n)}_{(\mathbf{a})}}$$

*is the normalization map.*

*Proof.* The morphism $K_{\mathbf{a}}$ is proper, finite and bimeromorphic onto its image. Since $X^{(\mathbf{a})}$ is a normal complex space, it is the normalization of $\overline{X^{(n)}_{(\mathbf{a})}}$. ∎

Define

$$K := \coprod_{\mathbf{a} \in P(n)} K_{\mathbf{a}} \colon \coprod_{\mathbf{a} \in P(n)} X^{(\mathbf{a})} \to X^{(n)},$$

and, for every integer $1 \leqslant l \leqslant n$,

$$K_l := \coprod_{\lambda(\mathbf{a})=l} K_{\mathbf{a}}\colon \coprod_{\lambda(\mathbf{a})=l} X^{(\mathbf{a})} \to \bigcup_{\lambda(\mathbf{a})=l} \overline{X^{(n)}_{(\mathbf{a})}} = \coprod_{\lambda(\mathbf{b})\leqslant l} X^{(n)}_{(\mathbf{b})} \subseteq X^{(n)}.$$

3.4. *The Sets* $F_z$, $S(\tilde{v})$, $F_z^{\mathbf{a}}$, $S^{\mathbf{a}}(\tilde{v})$, $F_z(l)$, *and* $S(l, \tilde{v})$

Let $z \in X^{(n)}$ be a point and $\tilde{v} := \tilde{v}(z)$ so that $z = \sum_j v_j x_j$ for pairwise distinct points $x_j \in X$. For every $\mathbf{a} \in P(n)$ and every integer $1 \leqslant l \leqslant n$ it is convenient to define

$$F_z := K^{-1}(z); \qquad F_z^{\mathbf{a}} := K_{\mathbf{a}}^{-1}(z); \qquad F_z(l) := \coprod_{\lambda(\mathbf{a})=n-l} F_z^{\mathbf{a}}.$$

The reason for the notational switch $l \to n-l$ is the following

LEMMA 3.4.1. *Let $h$ be a non-negative integer. Then*

$$Supp(R^{2h}\pi_* \mathbb{Q}_{X^{[n]}}) = \coprod_{\lambda(\mathbf{a})\leqslant n-h} X^{(n)}_{(\mathbf{a})}$$

(*this set is empty for $h \geqslant n$*),

$$Supp(R^{2h+1}\pi_* \mathbb{Q}_{X^{[n]}}) = \varnothing.$$

*Proof.* The assertion follows from Theorem 2.5.1.2, the local description of $\pi$ given in Subsection 3.5 and Künneth Formula. ∎

Let $\beta = (\mathbf{b_1}, ..., \mathbf{b_k}) \in \prod_{j=1}^k P(v_j)$ be a $k$-tuple of partitions in $a$-notation, and $\mathbf{b_j}(i)$ be the $i$th entry of the partition $\mathbf{b_j}$. Define a map

$$u\colon \prod_{j=1}^{k} P(v_j) \to P(n)$$

as follows: Let $\mathbf{b_j}(i) = 0$, if $i > v_j$ and assign to a $\beta$ as above, the partition $u(\beta)$ of $n$ which has $\sum_{j=1}^k \mathbf{b_j}(i)$ as $i$th entry in $a$-notation. Define, for every $0 \leqslant l \leqslant n-1$ and for every $\mathbf{a} \in P(n)$

$$S(\tilde{v}) := \prod_{j=1}^{k} P(v_j), \qquad S^{\mathbf{a}}(\tilde{v}) := u^{-1}(\mathbf{a}), \qquad S(l, \tilde{v}) := \coprod_{\lambda(\mathbf{a})=n-l} S^{\mathbf{a}}(\tilde{v}).$$

The set $F_z$ is the disjoint union of all sets $F_z^{\mathbf{a}}$. In order to determine all the sets above it is sufficient to determine $F_z^{\mathbf{a}}$, for every $\mathbf{a} \in P(n)$.

Note that $F_z^{\mathbf{a}} \neq \varnothing$ iff $\mathbf{a} \geqslant \tilde{v}$, i.e., iff $X^{(n)}_{(\tilde{v})} \subseteq \overline{X^{(n)}_{(\mathbf{a})}}$.

In order to determine $F_z^{\mathbf{a}}$ we need to impose the condition

$$K_{\mathbf{a}}(\tau)=\sum_{i=1}^{n} i(x_1^i+\cdots+x_{a_i}^i)=\sum_{j=1}^{k} \nu_j x_j.$$

It follows that

$$\tau=(\varepsilon_{11}x_1+\cdots+\varepsilon_{1k}x_k,\ldots,\varepsilon_{n1}x_1+\cdots+\varepsilon_{nk}x_k),$$

where the $\varepsilon$'s are integers subject to the conditions:

(1) $\varepsilon_{lm}\geqslant 0$, $\forall l, m$;

(2) for every $j=1, \ldots, k$, we have $\sum_{i=1}^{n} i\varepsilon_{ij}=\nu_j$;

to introduce the third condition, note that by (1) and (2) above, a solution $\varepsilon_{lm}$ gives an element $\beta=(\mathbf{b_1}, \ldots, \mathbf{b_k})\in S(\tilde{\nu})$: for every $j=1, \ldots, k$ the sequence $(\varepsilon_{1j}, \ldots, \varepsilon_{nj})$ is a partition of $\nu_j$ in $a$-notation; the third condition is that

(3) $\beta\in S^{\mathbf{a}}(\tilde{\nu})$.

There are natural bijections

$$F_z\Leftrightarrow S(\tilde{\nu}),\qquad F_z^{\mathbf{a}}\Leftrightarrow S^{\mathbf{a}}(\tilde{\nu}),\qquad F_z(l)\Leftrightarrow \coprod_{\lambda(\mathbf{a})=n-l} S^{\mathbf{a}}(\tilde{\nu})=S(l,\tilde{\nu}).$$

The elements of $F_z$ will be denoted by $\zeta_{z\beta}$, the ones of $F_z^{\mathbf{a}}$ by $\zeta_{z\beta}^{\mathbf{a}}$ and the ones of $F_z(l)$ by $\zeta_z(l)_\beta$.

Fix an integer $0\leqslant h\leqslant n-1$. The set $S(h,\tilde{\nu})$ can also be described as follows. Let $k:=\lambda(\tilde{\nu})$ and $H$ be the set of $k$-tuples $h=(h_1, \ldots, h_k)$ of non-negative integers with the property that $h=\sum h_j$. Then $S(h,\tilde{\nu})\subset\prod_{j=1}^{k} P(\nu_j)$ is the set of $k$-tuples $\beta=\{\mathbf{b_1}, \ldots, \mathbf{b_k}\}$ of partitions $\mathbf{b_j}\in P(\nu_j)$ with $\lambda(\mathbf{b_j})=\nu_j-h_j$ for some $(h_1, \ldots, h_k)\in H$.

### 3.5. *Local Model for the Douady–Barlet Morphism*

Let $z$ be a point in $X^{(n)}$. This point determines the unique stratum $X^{(n)}_{(\tilde{\nu}(z))}$ on which it lies. Let $\nu_1, \ldots, \nu_k$ be the entries of $\tilde{\nu}(z)$. Let $U_z(\varepsilon)=\prod_{j=1}^{k} {}_{x_j}\Delta^{(\nu_j)}(\varepsilon)$ be a basic neighborhood of $z$ in $X^{(n)}$. By abuse of notation, we shall write ${}_{x_j}\Delta^{[\nu_j]}(\varepsilon)$ in place of $({}_{x_j}\Delta(\varepsilon))^{[\nu_j]}$.

Around $z$, the Douady–Barlet morphism can and will be identified with the natural morphism

$$\prod_{j=1}^{k} {}_{x_j}\Delta^{[\nu_j]}(\varepsilon)\to\prod_{j=1}^{k} {}_{x_j}\Delta^{(\nu_j)}(\varepsilon).$$

In fact we can identify naturally the space on the left with the pre-image under $\pi$ of the space on the right.

### 3.6. *The Morphism of Complexes*

We first make the following elementary observation which simplifies the picture.

LEMMA 3.6.1. $\overline{X^{[n]}_{(\tilde\nu)}} = \pi^{-1}(\overline{X^{(n)}_{(\tilde\nu)}})$.

*Proof.* By the local description of the Douady–Barlet map it is enough to prove the following:

*Claim.* Let $\mathscr{I}$ be the ideal of a subscheme of $\Delta$ of length $m$ concentrated at $o \in \Delta$. Let $\tilde\mu = (\mu_1, ..., \mu_l)$ be a partition of $m$. There exists a family of subspaces in the stratum $X^{[m]}_{(\tilde\mu)}$ specializing to $\mathscr{I}$.

By virtue of Briançon's result Corollary 2.6.2, $\Delta^{[m]}_o$ is irreducible of dimension $m-1$, so that it is enough to prove the Claim for every $\mathscr{I}$ in a dense subset $U$ of $\Delta^{[m]}_o$. We choose $U$ to be the subset of curvilinear subspaces in $\Delta^{[m]}_o$. After a change of variables, such an ideal can be written as $\mathscr{I} = (y - c_1 x - \cdots - c_{n-1} x^{n-1}, x^n)$, $c_i \in \mathbb{C}$. The seeked for family is given by $(y - c_1 x - \cdots - c_{n-1} x^{n-1}, (x-\gamma_1)^{\mu_1} \cdots (x-\gamma_l)^{\mu_l})$, $\gamma_i \in \mathbb{C}$. ∎

As a consequence, $\pi_{|\overline{X^{[n]}_{(\tilde\nu)}}}: \overline{X^{[n]}_{(\tilde\nu)}} = \pi^{-1}(\overline{X^{(n)}_{(\tilde\nu)}}) \to \overline{X^{(n)}_{(\tilde\nu)}}$ has irreducible fibers and, for any $U \subseteq X^{(n)}$ and any partition $\tilde\nu$, the irreducible components of $X^{[n]}_{(\tilde\nu)} \cap \pi^{-1}(U)$ are in one to one correspondence with those of $X^{(n)}_{(\tilde\nu)} \cap U$. Note that if we did not prove Lemma 3.6.1 we would still have a natural injective correspondence between irreducible components on the target and on the source of $\pi$.

To perform our basic construction it is natural to use Borel–Moore homology. We recall a few basic facts about it (cf. [15, 19.1] and the references there).

(a) For an $n$-dimensional complex space $X$, a basis for $H^{lf}_{2n}(X, \mathbb{Q})$ is given by the irreducible components of $X$.

(b) $H^{lf}_\bullet$ is covariant with respect to proper morphisms, e.g., closed immersions.

(c) An open embedding $j: U \to X$ gives a restriction morphism

$$j^*: H^{lf}_\bullet(X, \mathbb{Q}) \to H^{lf}_\bullet(U, \mathbb{Q}).$$

Fix a partition $\tilde\nu$ of $n$ and consider the sheaf $\mathscr{F}^{[n]}_{\tilde\nu}$ on $X^{(n)}$ associated with the presheaf

$$\begin{aligned} U \to \{&\mathbb{Q}\text{-vector space generated by the irreducible} \\ &\text{components of } \pi^{-1}(U) \cap \overline{X^{[n]}_{(\tilde\nu)}}\} \\ = {}& H^{lf}_{2(n+\lambda(\tilde\nu))}(\overline{X^{[n]}_{(\tilde\nu)}} \cap \pi^{-1}(U), \mathbb{Q}), \end{aligned}$$

where the presheaf structure is given by the restriction morphisms stemming from (c).

The push-forward associated with the closed embedding $\overline{X^{[n]}_{(\tilde{\nu})}} \to X^{[n]}$ and Poincaré Duality on $X^{[n]}$ identify canonically $\mathscr{F}^{[n]}_{\tilde{\nu}}$ with a subsheaf $R^n_{\tilde{\nu}}$ of $R^{2(n-\lambda(\tilde{\nu}))}\pi_* \mathbb{Q}_{X^{[n]}}$.

From the previous discussion it follows that the presheaf is isomorphic to the analogous one defined by the irreducible components of $U \cap \overline{X^{(n)}_{(\tilde{\nu})}}$, which, by Zariski Main Theorem, is isomorphic to $K_{\mathbf{a}*}\mathbb{Q}_{X^{(\mathbf{a})}}$, where $\mathbf{a}$ is $\tilde{\nu}$ in $a$-notation.

Let $(\mathscr{D}^\bullet, d)$ be the resolution of the constant sheaf $\mathbb{C}_{X^{[n]}}$ given by the complex of currents. It is an acyclic resolution with respect to $\pi_*$. Let $\mathscr{Z}^\bullet$ denote the subcomplex of closed currents. Every analytic cycle defines the closed current of integration along itself. We can therefore define a morphism of sheaves $\Psi^{[n]}_{\tilde{\nu}}: \mathscr{F}^{[n]}_{\tilde{\nu}} \to \pi_* \mathscr{Z}^{2(n-\lambda(\tilde{\nu}))} \subseteq \pi_* \mathscr{D}^{2(n-\lambda(\tilde{\nu}))}$ whose projection on the cohomology sheaves $\mathscr{H}^{2(n-l(\nu))}(\pi_*\mathscr{D}^\bullet, d) = R^{2(n-\lambda(\tilde{\nu}))}\pi_*\mathbb{C}_{X^{(n)}}$ gives the previously defined identification with $R^n_{\tilde{\nu}}$.

We summarize the previous discussion in the following

Proposition 3.6.2. *There is a natural injective morphism of complexes of sheaves,*

$$\Psi: \bigoplus_{h=0}^{n-1} \bigoplus_{\lambda(\mathbf{a})=n-h} K_{\mathbf{a}*}\mathbb{C}_{X^{(\mathbf{a})}}[-2h] \to \pi_*\mathscr{D}^\bullet,$$

*where the l.h.s complex is endowed with zero differentials.*

Our main result is that $\Psi$ is a quasi-isomorphism. We shall need the following

Proposition 3.6.3. *Let $U$ be a basic neighborhood of a point $z \in X^{(n)}_{(\tilde{\nu})}$. A basis for $H^{2h}(\pi^{-1}(U), \mathbb{Q})$ is given by the cohomology classes of the irreducible components of $X^{[n]}_{(\tilde{\mu})} \cap \pi^{-1}(U)$, for all $\tilde{\mu} \leqslant \tilde{\nu}$ (i.e., $X^{[n]}_{(\tilde{\nu})} \subseteq \overline{X^{[n]}_{(\tilde{\mu})}}$) with $\lambda(\tilde{\mu}) = n-h$.*

*Proof.* Since $U$ is basic, we have $\pi^{-1}(U) \simeq \prod_{j=1}^k {}_{x_j}\Delta^{[\nu_j]}(\varepsilon)$ and

$$H^{2h}\left(\prod_{j=1}^k {}_{x_j}\Delta^{[\nu_j]}(\varepsilon), \mathbb{Q}\right) = \bigoplus_{\sum_j h_j = h;\, h_j \geqslant 0} \bigotimes_{j=1}^k H^{2h_j}({}_{x_j}\Delta^{[\nu_j]}(\varepsilon), \mathbb{Q}).$$

By virtue of this Künneth decomposition and of Theorem 2.5.1.3, we can form a basis for the vector space above by taking products of the subvarieties that we obtain via Nakajima's construction on each factor ${}_{x_j}\Delta^{[\nu_j]}$.

More precisely, we take the cohomology classes associated with the closed subvarieties

$$
{}_z\Delta^{[\tilde{\nu}]}_{\beta}(\varepsilon) := \left\{ \prod_{j=1}^{k} \overline{{}_{x_j}\Delta^{[\nu_j]}_{(\mathbf{b_j})}(\varepsilon)} \right\}
$$

which are indexed by the $k$-tuples of partitions $\beta = (\mathbf{b_1}, ..., \mathbf{b_k}) \in S(h, \tilde{\nu}) \subseteq \prod_{j=1}^{k} P(\nu_j)$ since

$$
{}_z\Delta^{[\tilde{\nu}]}_{\beta}(\varepsilon) = \overline{\pi^{-1}\left(\left\{ \prod_{j=1}^{k} {}_{x_j}\Delta^{(\nu_j)}_{(\mathbf{b_j})}(\varepsilon) \right\}\right)}.
$$

By virtue of the combinatorics previously developed in Subsection 3.4 and above, and because of Lemma 3.6.1, these are precisely the branches of $\overline{X^{[n]}_{(\tilde{\mu})}} \cap \pi^{-1}U$ for $\tilde{\mu} \leqslant \tilde{\nu}$ and $\lambda(\tilde{\mu}) = n - h$. ∎

Note that the complex spaces ${}_z\Delta^{(\tilde{\nu})}_{\beta}(\varepsilon)$ are irreducible in $U_z(\varepsilon)$, but not locally so: they may become locally reducible around certain points $y \in U_z(\varepsilon)$; in this case ${}_z\Delta^{(\tilde{\nu})}_{\beta}(\varepsilon)$ breaks up around $y$ into the union of its irreducible components.

EXAMPLE 3.6.4. Let $n = 4$, $z = 4z_1 \in X^{(4)}_{(4)}$, $y = 2y_1 + 2y_2 \in X^{(4)}_{(2,2)}$, where $y_1$ and $y_2$ are "near" $z_1$. The space $\overline{X^{(4)}_{(2,1,1)}}$ is locally irreducible around $z$: there is the only branch $\overline{{}_{z_1}\Delta^{(4)}_{(2,1,1)}(\varepsilon)}$ corresponding to $\beta = (1^2, 2^1)$. The closed subvariety $\overline{X^{(4)}_{(2,1,1)}}$ is locally reducible around $y$: there are two branches $\overline{{}_{y_1}\Delta^{(2)}_{(2)}(\varepsilon)} \times \overline{{}_{y_2}\Delta^{(2)}_{(1,1)}(\varepsilon)}$ and $\overline{{}_{y_1}\Delta^{(2)}_{(1,1)}(\varepsilon)} \times \overline{{}_{y_2}\Delta^{(2)}_{(2)}(\varepsilon)}$ and they correspond to $\beta_I = (2^1, 1^2)$ and $\beta_{II} = (1^2, 2^1)$, respectively.

## 4. DECOMPOSITION THEOREM FOR THE DOUADY–BARLET MORPHISM

### 4.1. *Proof of the Decomposition Theorem*

THEOREM 4.1.1. *Let $X$ be a complex surface. The injective morphism of complexes of Proposition* 3.6.2

$$
\Psi\colon \bigoplus_{h=0}^{n-1} \bigoplus_{\lambda(\mathbf{a}) = n-h} K_{\mathbf{a}*}\mathbb{C}_{X^{(\mathbf{a})}}[-2h] \to \pi_*\mathscr{D}^\bullet
$$

*is a quasi-isomorphism, i.e., it induces isomorphisms on the cohomology sheaves.*

*In particular,* $\mathbb{R}\pi_*\mathbb{C}_{X^{[n]}}$ *is isomorphic, in the derived category, to a complex with trivial differentials.*

*Proof.* Since the differentials of the complex on the left hand side, abbreviated by "*CL*," are trivial, we have

$$\mathscr{H}^t(CL) = \begin{cases} \displaystyle\bigoplus_{\lambda(\mathbf{a})=n-h} K_{\mathbf{a}*}\mathbb{C}_{X^{(\mathbf{a})}}, & \text{if} \quad t=2h \quad \text{and} \quad 0 \leqslant h \leqslant n-1, \\ 0, & \text{if otherwise.} \end{cases}$$

The cohomology sheaves of the complex on the right hand side *CR* are

$$\mathscr{H}^t(CR) = \begin{cases} Ker(\pi_* d^{2h})/Im(\pi_* d^{2h-1}) \simeq R^{2h}\pi_*\mathbb{C}_{X^{[n]}}, \\ \qquad \text{if} \quad t=2h \quad \text{and} \quad 0 \leqslant h \leqslant n-1, \\ 0, \qquad \text{if otherwise.} \end{cases}$$

The conclusion of the theorem is equivalent to showing that the induced map on the stalks $(\mathscr{H}^{2h}(\Psi))_z$ is bijective for every $z \in X^{(n)}$ and for every $0 \leqslant h \leqslant n-1$. This is precisely the content of Proposition 3.6.3. ∎

## 4.2. *Some Remarks*

*Remark* 4.2.1. The complex $\mathbb{R}\pi_*\mathbb{C}_{X^{[n]}}$ is defined up to isomorphism in the derived category of the category of complexes of sheaves on $X^{(n)}$. Note that the trivial complexes $K_{\mathbf{a}*}\mathbb{Q}_{X^{(\mathbf{a})}}$ are a natural realization of $IC^\bullet(\mathbb{Q}_{X^{(n)}_{(\mathbf{a})}})$. The morphism $\Psi$ of Theorem 4.1.1 is an explicit, injective and natural quasi-isomorphism of complexes of sheaves between a complex with trivial differentials and a natural realization of $\mathbb{R}\pi_*\mathbb{C}_{X^{[n]}}$. This is why we call Theorem 4.1.1 "The Decomposition Theorem."

*Remark* 4.2.2. The authors of [17] place themselves in the algebraic context in order to use the Decomposition Theorem in [4]. The restriction to the algebraic case is not necessary. Recall that the Douady–Barlet morphism is projective by virtue of Theorem 2.3.1. M. Saito [29], using his theory of mixed Hodge modules, has proved the necessary result in the analytic category for projective morphisms. This approach gives a not necessarily natural isomorphism in a derived category, whereas Theorem 4.1.1 gives an explicit quasi-isomorphism of complexes. Our approach bypasses the use of the deep decomposition theorems [4, 29].

*Remark* 4.2.3. In order to use Saito's Decomposition Theorem, one needs the irreducibility result of Briançon [6, V.3.3] to identify the intersection cohomology complexes occurring in Saito's Decomposition Theorem; see [28, Sects. 6.1 and 6.2]. Our proof of Theorem 4.1.1 does not depend in any essential way on Briançon's result; see Corollary 2.6.2.

*Remark* 4.2.4. Because of the previous remarks, our proof of Theorem 4.1.1 is different in spirit and in detail from the one outlined above. In addition, our proof of the Göttsche Formula given below, being based on Theorem 4.1.1, besides working also in the analytic context, is significantly different from the ones in the literature concerning algebraic surfaces. In particular, we have not used [10, Theorem 1.1.(iv)] as in [16, 7].

*Remark* 4.2.5. By taking a suitable resolution of $\mathbb{Q}_{X[n]}$, one can prove, in the same way, that there is an analogous *natural* decomposition for $\mathbb{R}\pi_*\mathbb{Q}_{X[n]}$. We do not need this fact and we omit the details.

## 5. CONSEQUENCES OF THE DECOMPOSITION THEOREM

### 5.1. *The Leray Spectral Sequence for the Pair* $(\pi, \mathbb{Q}_{X[n]})$

An immediate consequence of Theorem 4.1.1, is the following

COROLLARY 5.1.1. *For every integer h there are natural isomorphisms*

$$R^{2h}\pi_*\mathbb{Q}_{X[n]} \simeq K_{n-h_*}\mathbb{Q}_{\coprod_{\lambda(\mathbf{a})=n-h} X^{(\mathbf{a})}} \simeq \bigoplus_{\lambda(\mathbf{a})=n-h} K_{\mathbf{a}*}\mathbb{Q}_{X^{(\mathbf{a})}},$$

*and* $R^{2h+1}\pi_*\mathbb{Q}_{X[n]} = 0$.

*Proof.* For $\mathbb{C}$-coefficients it follows from Theorem 4.1.1 by taking cohomology sheaves.

For every $\mathbf{a} = \tilde{\nu} \in P(n)$, we have a natural identification of $K_{\mathbf{a}*}\mathbb{Q}_{X^{(\mathbf{a})}}$ with $R^n_{\tilde{\nu}}$ (see Subsection 3.6). The assertion for $\mathbb{Q}$-coefficients follows in view of Proposition 3.6.3. ∎

THEOREM 5.1.2. *The Leray spectral sequence for the pair* $(\pi, \mathbb{Q}_{X[n]})$ *is* $E_2$*-degenerate.*

*Proof.* In fact Theorem 4.1.1 and the finiteness of the morphisms $K_{\mathbf{a}}$ imply the following *natural* decomposition in the derived category,

$$\mathbb{R}\pi_*\mathbb{C}_{X[n]} \simeq \bigoplus_{j\geqslant 0} R^j\pi_*\mathbb{C}_{X[n]}[-j].$$

The $E_2$-degeneration for $\mathbb{C}$-coefficients implies the one for $\mathbb{Q}$-coefficients. ∎

*Remark* 5.1.3. In view of Remark 4.2.2, one sees that a decomposition as in Theorem 5.1.2 holds with $\mathbb{Q}$-coefficients, however, one may prefer a natural one. One gets one by virtue of Remark 4.2.5.

## 5.2. *Göttsche's Formula and Nakajima's Interpretation*

THEOREM 5.2.1 (Göttsche's Formula). *Let $X$ be a complex surface. Then for every $l \in \mathbb{Z}$ and every integer $n \in \mathbb{N}$, we have a natural isomorphism,*

$$H^l(X^{[n]}, \mathbb{Q}) \simeq \bigoplus_{\mathbf{a} \in P(n)} H^{l-2n+2\lambda(\mathbf{a})}(X^{(\mathbf{a})}, \mathbb{Q}).$$

*Assume in addition that $X$ has finite Betti numbers $b_i(X)$, $i = 0, ..., 4$. Then the Douady spaces $X^{[n]}$ have finite Betti numbers and the generating function for the Poincaré polynomials is*

$$\sum_{n=0}^{\infty} P_t(X^{[n]})\, q^n = \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1(X)}\,(1+t^{2m+1}q^m)^{b_3(X)})}{(1-t^{2m-2}q^m)^{b_0(X)}\,(1-t^{2m}q^m)^{b_2(X)}\,(1-t^{2m+2}q^m)^{b_4(X)}}. \tag{2}$$

*Proof.* The quasi-isomorphism of Theorem 4.1.1 induces an isomorphism in hypercohomology. This proves the first assertion for $\mathbb{C}$-coefficients. The assertion for $\mathbb{Q}$-coefficients follows from Corollary 5.1.1 and Theorem 5.1.2 (cf. also Remark 4.2.5). Formula (2) follows from a formal manipulation which builds on Macdonald's Formula; see [28, p. 69]. ∎

*Remark* 5.2.2. Theorem 5.2 is new for $X$ non algebraic. If $X$ is algebraic, then this result is slightly more precise than the corresponding statement in [17] since it distinguishes a natural isomorphism.

The following corollary has been proved independently by the first author using elementary topology in [8]. The r.h.s. exhibits modular behavior [16].

COROLLARY 5.2.3 (cf. [8]). *Let $X$ be a complex surface with finite Betti numbers. Let $e(X^{[n]})$ be the Euler number of $X^{[n]}$. We have the following generating function for $e(X^{[n]})$:*

$$\sum_{n=0}^{\infty} e(X^{[n]})\, q^n = \prod_{m=1}^{\infty} \left(\frac{1}{1-q^m}\right)^{e(X)}. \tag{3}$$

*Proof.* Set $t = -1$ in Göttsche's formula Theorem 5.2.1. ∎

THEOREM 5.2.4 (Nakajima's Interpretation of Göttsche's Formula). *Let $X$ be a complex surface with finite Betti numbers. The super vector space $\mathbb{H}(X) = \bigoplus_{n \geq 0} H^*(X^{[n]}, \mathbb{Q})$ is an irreducible highest weight representation* *as the representation of the Heisenberg/Clifford algebra, with highest weight vector the generator* $\mathbf{1} \in H^0(X^{[0]}, \mathbb{Q}) \simeq \mathbb{Q}$.

*Proof.* By virtue of Theorem 2.4.4, Theorem 5.2.1 was the only missing piece. ∎

*Remark* 5.2.5. For a geometric action of the Virasoro algebra see [26].

## 5.3. *The Hodge Structure in the Kähler Case*

For the following theorem recall that the cohomology of the quotient by a finite group of a smooth compact Kähler manifold carries a pure Hodge structure.

THEOREM 5.3.1. *Let $Y$ be a compact, Kähler, complex surface. Let $X$ be a Zariski-dense open subset of $Y$. Then for every $l \in \mathbb{Z}$ and every $n \in \mathbb{N}^+$ we have a natural isomorphism of mixed Hodge structures*:

$$H^l(X^{[n]}) \otimes \mathbb{Q}(n) \simeq \bigoplus_{\mathbf{a} \in P(n)} H^{l-2n+2\lambda(\mathbf{a})}(X^{(\mathbf{a})}, \mathbb{Q}) \otimes \mathbb{Q}(\lambda(\mathbf{a})).$$

*Proof.* Note that $X$ has finite Betti numbers. Every class $\alpha \in H^*(X^{[n]})$ can be represented as $P_{\alpha_1}[i_1] \circ \cdots \circ P_{\alpha_r}[i_r](\mathbf{1})$. If the classes $\alpha_i$ have type $(p_i, q_i)$, then $\alpha$ has type $(\sum p_i + n - l(\tilde{\nu}), \sum q_i + n - l(\tilde{\nu}))$. The statement now follows after a formal manipulation. ∎

*Remark* 5.3.2. In the quasi-projective case Theorem 5.3.1 (without the naturality assertion) was first proved in [17] using Saito's Theory of mixed Hodge modules. The remark that a proof in the projective case depending on Nakajima's operators is possible can be found in [18] and has also been made by Nakajima in a private communications to us.

Note that one can compute the generating function for the virtual Hodge numbers as in [7].

## 5.4. *Connection with Equivariant K-Theory*

In this section we prove Theorem 5.4.3.

Our original motivation was to explain the following three sets of equalities by means of the existence of *natural* isomorphisms. Let $G \subseteq U(2)$ be a finite subgroup, $\mathbb{C}^2/G$ be the associated "Kleinian singularity" and $Y$ be its associated minimal resolution of singularities.

- The papers [1, 20] remark that the "orbifold Euler number" $e(\mathbb{C}^2, G)$ and the Euler number $e(Y)$ coincide.

- The paper [20] remarks that the equality $e(X^n, \mathfrak{S}_n) = e(X^{[n]})$ holds for any smooth algebraic surface $X$.

• The paper [33] remarks that $\dim_{\mathbb{C}} K_{\mathfrak{S}_n}(Y^n) \otimes_{\mathbb{Z}} \mathbb{C} = \dim_{\mathbb{C}} K(Y^{[n]}) \otimes_{\mathbb{Z}} \mathbb{C}$. The same is true for every surface $Y$.

We explain these equalities via the natural isomorphism in Theorem 5.4.3.

Let us recall some notions and facts relating them.

Let $G$ be a finite group and $Y$ be a locally compact, Hausdorff and paracompact left $G$-space.

Denote by $K_G(Y)$ the Equivariant $K$-Theory of the pair $(Y, G)$; see [1, 3, 33]. It is a $\mathbb{Z}_2$-graded abelian group. Its "even" part is the Grothendieck group generated by $G$-vector bundles. We shall not consider the multiplicative structure induced by the tensor product.

Let $a \in G$ be an element and define $Y^a := \{ y \in Y \mid ya = y \}$.

Define $\hat{Y} := \{ (y, b) \in Y \times G,\ yb = y \}$. There is a natural identification $\hat{Y} = \coprod_{g \in G} Y^g$. There is a natural $G$-action of $G$ on $\hat{Y}$ given by $(y, c)\, d := (yd, d^{-1}cd)$.

Let $G_*$ be the set of conjugacy classes of $G$, $g \in G$ be an element, $[g]$ be its conjugacy class, and $Z_G(g)$ be the centralizer of $g$ in $G$; this subgroup acts on $Y^g$. If $[g] = [g']$, then there is a canonical identification $Y^g/Z_G(g) \simeq Y^{g'}/Z_G(g')$.

Choose representatives $\mathbf{g} = \{ g_1, ..., g_{|G_*|} \}$ for each conjugacy class in $G_*$.

There is a homeomorphism $\alpha_{\mathbf{g}}\colon \hat{Y}/G \simeq \coprod_{[g] \in G_*} Y^{g_l}/Z_G(g_l)$.

The following relates the $G$-Equivariant $K$-Theory of $Y$ to the $K$-Theory of the fixed-point-sets.

THEOREM 5.4.1 (cf. [1, 3]). *Let $G$, $Y$, $\hat{Y}$, $\mathbf{g}$, and $\alpha_{\mathbf{g}}$ be as above. There are $\mathbb{Z}_2$-graded isomorphisms of $\mathbb{C}$-vector spaces,*

$$\phi_{\mathbf{g}}\colon K_G(Y) \otimes_{\mathbb{Z}} \mathbb{C} \to K(\hat{Y}/G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\alpha_{\mathbf{g}}^{-1}} \bigoplus_{l=1}^{|G_*|} K(Y^{g_l}/Z_G(g_l)) \otimes_{\mathbb{Z}} \mathbb{C},$$

*where the first one is natural and the second one depends on* $\mathbf{g}$.

*Remark* 5.4.2. If $G = \mathfrak{S}_n$, then Theorem 5.4.1 holds with $\mathbb{Q}$-coefficients.

THEOREM 5.4.3 (Connection with Equivariant $K$-Theory). *Let $X$ be a complex surface. For every natural integer $n$ there is a natural $\mathbb{Z}_2$-graded $\mathbb{Q}$-linear isomorphisms,*

$$K_{\mathfrak{S}_n}(X^n) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq K(X^{[n]}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

*Proof.* Let $Y := X^n$ and $G := \mathfrak{S}_n$, where the action is given by the permutation of the factors. There is a natural and well-known identification $P(n) = \mathfrak{S}_{n*}$. Having made a choice $\mathbf{g}$ as above we obtain, as in [20], a natural identification $\beta\colon \hat{Y}/\mathfrak{S}_n \simeq \coprod_{\mathbf{a} \in P(n)} X^{(\mathbf{a})}$ which does not depend on $\mathbf{g}$.

By virtue of Theorem 5.4.1, of the existence of $\beta$ and of Theorem 5.2.1, we get a natural $(\mathbb{Z}/2\mathbb{Z})$-graded isomorphism of graded $\mathbb{Q}$-vector spaces,

$$K_{\mathfrak{S}_n}(X^n)\otimes_{\mathbb{Z}}\mathbb{Q}\simeq K(\hat{Y}/\mathfrak{S}_n)\otimes_{\mathbb{Z}}\mathbb{Q}\simeq\bigoplus_{\mathbf{a}\in P(n)}K(X^{(\mathbf{a})})\otimes_{\mathbb{Z}}\mathbb{Q}\simeq K(X^{[n]})\otimes_{\mathbb{Z}}\mathbb{Q},$$

where the last isomorphism stems from Theorem 5.2.1 after taking the Chern Character isomorphism: $K(-)\otimes_{\mathbb{Z}}\mathbb{Q}\simeq H^*(-,\mathbb{Q})$. ∎

*Remark* 5.4.4. After we proved Theorem 5.4.3, we received a copy of [5] which contains a statement similar to the one of Theorem 5.4.3, but where the natural map proposed in [5] should be constructed in an entirely different way. We thank V. Ginzburg for giving us a copy of [5].

*Remark* 5.4.5. In a lecture at Cambridge, G. Segal [30] (see also [33]) introduced new structures on the vector space $\mathbb{K}(X):=\bigoplus_{n\geqslant 0}K_{\mathfrak{S}_n}(X^n)\otimes_{\mathbb{Z}}\mathbb{Q}$.

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