Math 322 Lecture 1 (Tu/Th) FINAL SPRING 2009 Mark de Cataldo

1. (35 total pts) State and prove the Heine-Borel theorem for $[0, 1]$.

Heine-Borel; $[0,1]$ is compact.

Consider the set : $S = \{x \in [0,1] \mid [0,x]$ can be covered by a finite cover$\}$.

$0 \in S \Rightarrow S \neq \emptyset$, bounded above $\Rightarrow S$ has a supremum.

let $b' = \sup S$. $\Rightarrow$

There is an element of the original cover, V, s.t $b' \in V$.

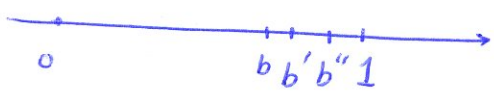
As V is open, $\exists\, b < b'$, $b \in V$. $\Rightarrow b \in S$

$\Rightarrow [0,b]$ can be covered by a finite cover, ω.

$\Rightarrow \omega \cup \{V\}$ covers $[0,b']$. But if $b' < 1$,

$\exists\, b'' > b'$, $b'' < 1$, s.t $b'' \in V$. $\Rightarrow [0,b'']$ can be covered by $\omega \cup \{V\}$, which is finite. Contradiction.

$\rightarrow b' = 1$ and $b' \in S \Rightarrow S = [0,1]$.



2. (35 total pts) State and prove the implicit function theorem (you may use the inverse function theorem without stating it, nor proving it).

Implicit function thm:

If $f: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, is continuously differentiable in the open set Ω, and $(x_0, \underbrace{y_0}_{\in \mathbb{R}^m}) \in \Omega$ ($x_0 \in \mathbb{R}^n$), and $f(x_0, y_0) = 0$, and the ~~left~~ right-most $m \times m$ submatrix of Df is invertible $\rightarrow$ There is an open rectangle $A \times B$, $(x_0, y_0) \in A \times B$, such that $\exists g$: differentiable : $y = g(x)$ ~~on $A \times B$~~, and $y_0 = g(x_0)$.

Proof; Define: $\begin{cases} F: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m \\ (x, y) \longmapsto (x, f(x,y)) \end{cases} \longrightarrow DF = \left[\begin{array}{c|c} I_{n \times n} & 0 \\ \hline \multicolumn{2}{c}{Df} \end{array} \right]$

$\longrightarrow DF$ is invertible. By inverse function theorem, $\exists$ open rectangle $A \times B$ on which $F(\cdot,\cdot)$ has got a continuously differentiable inverse: $\begin{cases} G: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m \\ (x,y) \longmapsto (k(x,y), \tilde{g}(x,y)) \end{cases}$

But $\underbrace{F \circ G}_{id}(x,y) = (k(x,y), f(k(x), \tilde{g}(x,y))) \longrightarrow k(x,y) = x$

$\rightarrow f(x, \tilde{g}(x,y)) = y$ let $y = 0 \longrightarrow$

$f(x, \tilde{g}(x,0)) = 0$ so define $g(x) := \tilde{g}(x,0)$. ∎

3. (35 total pts) Let ω be a differential k-form on R^n. Prove that $d^2\omega = 0$.

We know that the operator 'd' is linear, that is $d(\omega + \eta) = d\omega + d\eta$. Also, we know that every k-form is a ~~linear~~ summation over $f_{i_1,\dots,i_k}\, dx^{i_1}\wedge \dots \wedge dx^{i_k}$'s. So it suffices to prove that $d^2(f\, dx^1\wedge \dots \wedge dx^k) = 0$

$$d(f\, dx^1\wedge \dots \wedge dx^k) = \sum_{1\le j\le n} \left(\frac{\partial f}{\partial x^j} dx^j\right)\wedge dx^1\wedge \dots \wedge dx^k.$$

$$d^2(f\, dx^1\wedge \dots \wedge dx^k) = \sum_{1\le i\le n} \sum_{1\le j\le n} \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j} f\right) dx^i\wedge dx^j\wedge dx^1\wedge \dots \wedge dx^k.$$

But note that $\forall i,j$, $\begin{cases} \dfrac{\partial^2}{\partial x^i\partial x^j} f = \dfrac{\partial^2}{\partial x^j\partial x^i} f \\ dx^i\wedge dx^j = -dx^j\wedge dx^i \end{cases}$ $\rightarrow$.

$$\frac{\partial^2}{\partial x^i\partial x^j} f\, dx^i\wedge dx^j = -\frac{\partial^2}{\partial x^i\partial x^j} f\, dx^j\wedge dx^i.$$

$\rightarrow d^2(f\, dx^1\wedge \dots \wedge dx^k) = 0.$

4. (35 total pts) Let $f : R^n \to R$ be a function such that $|f(x)| \leq \frac{\pi}{e}|x|^{\sqrt{5}}$. Show that f is differentiable at $0 \in R^n$.

let $r \in \mathbb{R}$ be s.t $r > 1$. Also, $f : \mathbb{R}^n \to \mathbb{R}$ be s.t $|f(x)| \leq C|x|^r$, for some $C > 0$.

We claim such an $f(\cdot)$ is differentiable at '0', with $Df|_0 = 0$

$$\frac{|f(x) - f(0)|}{|x|} = \frac{|f(x)|}{|x|} \leq C\frac{|x|^r}{|x|} = C|x|^{r-1} \longrightarrow 0 \text{ as } |x| \to 0$$

Finally, note that $\sqrt{5} > 1$.

5. (35 total pts) Find the derivative f' of the cross product function $f : R^3 \times R^3 \to R^3$.

$$f : (x_1, y_1, z_1, x_2, y_2, z_2) = (y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,$$
$$x_1 y_2 - x_2 y_1)$$

$$\to Df = \begin{bmatrix} 0 & z_2 & -y_2 & 0 & -z_1 & y_1 \\ -z_2 & 0 & x_2 & z_1 & 0 & -x_1 \\ y_2 & -x_2 & 0 & -y_1 & x_1 & 0 \end{bmatrix}$$

6. (35 total pts) Give an example of a function $R^2 \to R$ which admits all directional derivatives at $0 \in R^2$, but is not differentiable at $0 \in R^2$. (Justify all steps.)

Define $f(x,y) = x^{2/3} \cdot y^{1/3}$.

$\to$ at point '0': $\begin{cases} D_1 f(x,y)\big|_0 = 0 \\ D_2 f(x,y)\big|_0 = 0 \end{cases} \to$

As the directional derivatives vanish on two ~~at~~ axes, Df, (Total derivative) is zero provided that it exists.

But along the line $y = mx$ the directional derivative is $\cancel{\frac{\sqrt{m}}{1+m^2}}$ $\dfrac{\sqrt[3]{m}}{\sqrt{1+m^2}}$.

which is not zero. $\longrightarrow Df$ does not exist at '0'.

7. (35 total pts) Let $f : R^2 \to R$. Give a sufficient condition that implies $D_{12}f = D_{21}f$ and prove that your condition is sufficient. (Hint: Fubini's theorem.)

It is ~~then~~ sufficient that $D_{12}f$ & $D_{21}f$ be ~~are~~ continuous ~~integrable!~~ ~~for which~~ ~~it suffices for them to be continuous.~~

If D_{12} & D_{21} are cts $\longrightarrow$ they are integrable

By continuity, if at a point 'a' $D_{12}f - D_{21}f > 0$ (or similarly < 0), $\exists$ open rectangle R containing 'a' such that. on R: $D_{12}f - D_{21}f > 0$

$$\rightarrow \int_R (D_{12} - D_{21})f > 0 \qquad R = [a_1, b_1] \times [a_2, b_2].$$

But by Fubini's Thm: $\int_R D_{12}f = \int_{a_2}^{b_2} \int_{a_1}^{b_1} D_{12}f$

$$= \int_{a_2}^{b_2} D_2 f = f(b_1, b_2) - f(b_1, a_1) + f(a_1, a_2) - f(a_1, b_2)$$

and $\int_R D_{21}f = f(b_1, b_2) - f(b_1, a_1) + f(a_1, a_2) - f(a_1, b_2)$

Contradiction.

8. (35 total pts)

(a) State a necessary and sufficient condition for two basis in R^n to yield the same orientation.

Let $B_1\{v_1, \ldots, v_n\}$ and $B_1\{w_1, \ldots, w_n\}$ be two basis for $\mathbb{R}^n$ with $\{\phi_1, \ldots, \phi_n\}$ be the dual for $\{v_1, \ldots, v_n\}$.

B_1 & B_2 have the same orientation $\leftrightarrow$

$\phi_1 \wedge \cdots \wedge \phi_n (w_1, \ldots, w_n) > 0$

(b) Let $c : [0,1] \to (R^n)^n$ be continuous. Assume that for every $t \in [0,1]$, the n-tuple $(c_1(t), \ldots, c_n(t))$ of vectors in R^n is a basis of R^n.
Prove that the orientation $[c_1(0), \ldots, c_n(0)]$ equals the orientation $[c_1(1), \ldots, c_n(1)]$.

as $(c_1(t), \ldots, c_n(t))$ are linearly independent,

$$\underbrace{\det \begin{bmatrix} c_1^1(t) & \cdots & c_1^n(t) \\ \vdots & & \\ c_n^1(t) & \cdots & c_n^n(t) \end{bmatrix}}_{} \neq 0$$

$\to \phi(t)$

$\phi(t)$ keeps positive or negative (by continuity).

~~But~~ That is sign $\det[c_i^j(0)]$ = sign $\det[c_i^j(1)]$.

$\to$ They have the orientation

9. (35 total pts) Let F be the vector field on R^3 defined by the three functions:

$$F^1(x,y,z) = \sin(xyz), \qquad F^2(x,y,z) = \cos(x+y+z), \qquad F^3 = e^{xyz}.$$

Verify directly that

(a) $\operatorname{curl}\operatorname{grad} F^1 = 0$.

(b) $\operatorname{div}\operatorname{curl} F = 0$.

$$\vec{\nabla} F^1 = \overbrace{\cos(xyz)(yz, xz, xy)}^{G}.$$

$$\vec{\nabla}\times(\vec{\nabla}F) = \left(\frac{\partial}{\partial y}G^3 - \frac{\partial}{\partial z}G^2, \frac{\partial}{\partial z}G^1 - \frac{\partial}{\partial x}G^3, \frac{\partial}{\partial x}G^2 - \frac{\partial}{\partial y}G^1\right)$$

$$= \Big(\underbrace{\cancel{x\cos(xyz)} - \cancel{xz\sin(xyz)\,xy} - \cancel{x\cos(xyz)} + \cancel{xy\cdot xz\sin(xyz)}}_{0},$$

$$\ldots,$$

$$\ldots\Big)$$

similarly the second and 3rd component.

$$\nabla\times F = \left(\frac{\partial}{\partial y}F^3 - \frac{\partial}{\partial z}F^2, \frac{\partial}{\partial z}F^1 - \frac{\partial}{\partial x}F^3, \frac{\partial}{\partial x}F^2 - \frac{\partial}{\partial y}F^1\right)$$

$$= \Big(xze^{xyz} - \sin(x+y+z),\ xy\cos(xyz) - yze^{xyz},\ -\sin(x+y+z) - xz\cos(xyz)\Big)$$

$$\nabla\cdot(\nabla\times F) = \cancel{ze^{xyz}} + x^2yz^2e^{xyz} \mp \cancel{\cos(x+y+z)} +$$

$$\cancel{x\cos(xyz)} \mp \cancel{x^2yz\sin(xyz)} - \cancel{ze^{xyz}} - yz\cdot xze^{xyz} +$$

$$+\cancel{\cos(x+y+z)} - \cancel{x\cos(xyz)} + \cancel{x^2yz\sin(xyz)} = 0$$

10. (35 total pts) Let $I^3 : [0,1]^3 \to R^3$ be the standard singular cube. Assume Stoke's theorem and that $d^2 = 0$. Deduce from these two assumptions that $\partial^2 I^3 = 0$. (Do not prove the conclusion directly, you will get no credit for that).

By Stokes's Thm :

$$\int_{I^3} \underbrace{d^2\omega}_{0} = \int_{\partial I^3} d\omega = \int_{\partial^2 I^3} \omega$$

$$\rightarrow \int_{\partial^2 I^3} \omega = 0 \quad \cancel{\text{for any } \omega,\ k\text{-form}}$$

for any 1-form ω.

If $\partial^2 I^3 \neq 0$, it should contain at least one of the boundary 1-cubes. Without loss of ~~Define~~ generality let it be ~~$I^3(t,$~~ $\sigma = I^3(t, 0, 0)$, and define $\omega = dx^1 \cdot \phi$

$\rightarrow$ ~~$\int_\sigma \omega \neq \int_{\partial^2 I^3} \omega$~~ as Where ϕ is a non-negative smooth function vanishing outside the ball $B((1/2, 0, 0), 1/4)$.

$$\rightarrow \int_\sigma \omega = \int_{\partial I^3} \omega > 0 \quad \text{Contradiction.}$$

as ω is zero on other possible components of $\partial^2 I^3$.