- MAT 322
HW4.

2-29) a) Just the definition of partial derivative (cf. p.25).

b) $$\frac{f(a+h(tx))-f(a)}{h} = \underbrace{\frac{f(a+htx))-f(a)}{ht}}_{\to D_x f} \cdot t$$

c) Recall the limit in the definition of derivative:

If the limit $D_x f(a) = \lim_{t\to 0} \frac{f(a+tx)-f(a)}{t}$ (I) exists and

$$\lim_{t\to 0} \frac{\|f(a+tx)-f(a) - Df(a)(tx)\|}{\|tx\|} = 0 \quad \text{(II)} \quad \text{(differentiability)}.$$

then by uniqueness of limits, $D_x f(a) = Df(a)(x)$

Note: You can re-write

(I) as $$\lim_{t\to 0} \frac{f(a+tx)-f(a) - D_x f(a)\cdot t}{t} = 0$$

2-32) a) We've proved already, in the exercises, that as

$|f(x)| \le x^2$, $f(\cdot)$ is differentiable at '0'.

$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$, $x \neq 0$.

Evidently, $f'(x)$ is not conti. at '0'.

b) $$\frac{\|f(x,y)-f(0,0)\|}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}\,\sin\frac{1}{\sqrt{x^2+y^2}} = \|(x,y)\|\cdot \sin\frac{1}{\|(x,y)\|} \longrightarrow 0$$ as $\|(x,y)\| \longrightarrow 0$

So $Df = 0$.

$$D_1 f(x,y) = 2x \sin\frac{1}{\sqrt{x^2+y^2}} - \frac{x}{2}\,\frac{x^2+y^2}{(x^2+y^2)^{3/2}}\cos\frac{1}{\sqrt{x^2+y^2}}$$

$$= 2x \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{2}\,\frac{1}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Take two eq. $(1/n, 0)$, $(-1/n, 0)$ and they give different limits

2-33). $$\frac{|f(a+h)-f(a)-\sum_{i=1}^{n} D_i f(a)\cdot h^i|}{|h|} \le \frac{|f(a+h^1, a^2, .., a^n)-f(a)-D_1 f(a)\cdot h^1|}{|h|} \quad \text{I}$$

$$+\frac{|\sum_{i=2}^{n}[D_i f(c_i)-D_i f(a)]h^i|}{|h|} \quad \text{II}$$

$D_1 f(a)$ exists

$$|h| \ge |h^1| \rightarrow \text{I} \le \frac{|f(a+h^1, .., a^n)-f(a)-D_1 f(a)\cdot h^1|}{|h^1|} \rightarrow 0 \text{ as } h^1 \rightarrow 0.$$

II With the same argument as 2-8, (continuity), goes to '0'.

Note; Notation is the same as thm 2-8 and its proof.

2-35;

Following the hint, let $h_x(t) := f(tx)$. Then $h_x(t): \mathbb{R} \to \mathbb{R}$.

is a differentiable function. By chain rule:

$$\frac{d}{dt} h_x(t) = \frac{d}{dt}(tx) \cdot Df\Big|_{(tx)} = \underset{(x^1, .., x^n)}{\underset{\sim}{x}} \cdot Df\Big|_{tx}$$

$\Rightarrow h(1) = f(x)$, $h(0) = f(0) = 0$

$$\Rightarrow h(1) - h(0) = h(1) = f(x) = \int_0^1 \frac{d}{dt} h_x(t)\, dt =$$

$$x \cdot \int_0^1 Df\Big|_{tx}\, dt \quad (*)$$

– Df is a vector $\longrightarrow$ by definition, $\int Df$ is given by integrating componentwise. Let $g^i = \pi^i\left(\int_0^1 Df\big|_{tx}\, dt\right)$

$$= \int_0^1 \pi^i \circ Df\big|_{tx}\, dt$$

Then, by $(*)$ $f(x) = x^1 g^1 + \cdots + x^n g^n$

$\rightarrow$

2-36; Recall the statement of IFT. (Inverse Function Thm)

Let E be any <u>open</u> set on which $\det f'(x) \neq 0$, $x \in E$.

By IFT, $\forall x \in E$, $\exists V_x, W_x$ (open), $\begin{cases} x \in V \\ f(x) \in W \end{cases}$, such that $f(V) = W$. That is every point $y \in f(E)$ is contained in a open neighbhd W, s.t $W \subseteq f(E)$.

$\Rightarrow f(E) = \bigcup_{x \in E} W_x$ a union of open sets is open $\rightarrow$ $f(E)$ is open.

HW 4+3)

In special case, in this problem $f(A)$ is open. as well as $f(B)$.

To prove differentiability of $f^{-1}(\cdot)$, note that the IFT gives us the differentiability in all neighbhds like W (introduced above). $\Rightarrow f^{-1}$ is differentiable on $f(A)$.

2-37. Once we prove 'a', 'b' is proved with the same idea.

If $D_1 f(x,y) = 0$ in an open neighbhd, obviously in that neghbhd $f(x_1, y) = f(x_2, y)$. *

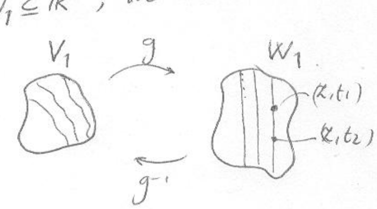
If $D_1 f(x_0, y_0) \neq 0 \rightarrow$ Exists an open neighbhd V, $x_0, y_0 \in V$, s.t $D_1 f(x,y) \neq 0$, $(x,y) \in V \subseteq \mathbb{R}^2$ (Why?).

Define $\begin{cases} g : V \to \mathbb{R}^2 \\ g(x,y) = (f(x,y), y) \end{cases} \Rightarrow Dg = \begin{pmatrix} D_1 f & D_2 f \\ 0 & 1 \end{pmatrix}$.

$\rightarrow \det Dg(x,y) \neq 0$, $(x,y) \in V$.

IFT $\Rightarrow$ For two neighbourhoods $V_1 \subseteq V$, $W_1 \subseteq \mathbb{R}^2$, we have

$\begin{cases} g^{-1} \circ g = id : V_1 \to V_1 \\ g \circ g^{-1} = id : W_1 \to W_1 \end{cases}$



Therefore, for two points

$(z, t_1), (z, t_2) \in W_1$

correspond $g^{-1}(z, t_1) \neq g^{-1}(z, t_2)$. $\Rightarrow$

$\downarrow \hookrightarrow \in V_1$ $\quad \downarrow \hookrightarrow \in V_1$

(x_1, y_1) $\quad (x_2, y_2)$

But $f(x_1, y_1) = f(x_2, y_2) \Rightarrow f(\cdot)$ cannot be one-one.

2-39; It ~~can~~ be easily checked that $f'(0) = \frac{1}{2} \neq 0$, [although

$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(\frac{1}{2} + 2x \sin \frac{1}{x} - \cos(\frac{1}{x})\right)$ does not exist].

But every neighbhd of the origin contains infinitely many pairs of points x, y : s.t $f(x) = f(y)$.

To see why, note that ~~at~~ in every nghbhd V of the origin, $f'(t) = 0$, at infinitely many points.

You can easily check that for all these points either $f''(t) > 0$ or $f''(t) < 0$ $\rightarrow$ they are local maxima or minima: $\frown$ $\smile$

$\longrightarrow f$ is not 1-1.