MAT 322; Analysis in Several Variables; HW3.

2-11; a) $f(x,y) = \int_a^{q(x,y)} g(t)\,dt$ $\qquad q(x,y) = x+y$.

$\rightarrow f'(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$.

$\frac{\partial f}{\partial x} = \frac{\partial q}{\partial x} \cdot \frac{d}{dq}\int_a^q g = 1 \cdot g(q(x,y)) = g(x+y)$.

$\frac{\partial f}{\partial y} = g(x+y)$.

b) $f'(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ $\qquad q(x,y) = xy$.

$\frac{\partial f}{\partial x} = y \cdot g(x+y)$.

$\frac{\partial f}{\partial y} = x \cdot g(x+y)$

c) $\frac{\partial f}{\partial x} = \frac{\partial q}{\partial x} \cdot g(q)$. $\qquad q = \sin(x\sin(y\sin z))$

$$\begin{cases} \frac{\partial q}{\partial x} = \sin(y\sin z) \cdot \cos(x\sin(y\sin z)) & \text{Chain-rule} \\ \frac{\partial q}{\partial y} = x\cdot\sin z \cdot \cos(y\sin z)\ \cos(x\sin(y\sin z)) \\ \frac{\partial q}{\partial z} = x \cdot y\cos z \cdot \cos(y\sin z) \cdot \cos(x\sin(y\sin z)) \end{cases}$$

$\frac{\partial f}{\partial y} = \frac{\partial q}{\partial y} g(q)$

$\frac{\partial f}{\partial z} = \frac{\partial q}{\partial z} g(q)$

2-12; a) Recall that $\begin{cases} h = \sum h_i \hat{e}_i \\ k = \sum k_j \hat{f}_j \end{cases}$ $\quad \{\hat{e}_i\}_i$: basis for $\mathbb{R}^n$, $\{\hat{f}_j\}_j$: basis for $\mathbb{R}^m$

$\rightarrow f(h,k) = \sum_i h_i f(\hat{e}_i, k) = \sum_i h_i \left[\sum_j f(\hat{e}_i, \hat{f}_j) k_j\right]$

$= \sum_{i,j} h_i k_j f(\hat{e}_i, \hat{f}_j)$

2-12. a, Cont'd:

The number of $f(\hat{e}_i, \hat{f}_j)$'s is $m \times n$. $\rightarrow$ finite $\rightarrow$ There is a maximum M for $|f(\hat{e}_i, \hat{f}_j)|$.

$\rightarrow$ Triangle inequality: $|f(h,k)| \leq M \sum_{i,j} |h_i k_j|$

Note that; $\sum |a_i| \leq n \max_i |a_i| \leq n \sqrt{\sum |a_i|^2}$ $*$.

Therefore: $|f(h,k)| \leq M \cdot \sum_i |h_i| \sum_j |k_j|$ (Re-~~writti~~ writing).

by $*$: $\leq M n^2 \underbrace{|h||k|}_{\text{Euclidean Norm.}}$

So: $\dfrac{|f(h,k)|}{|(h,k)|} \leq M n^2 \cdot \dfrac{|h||k|}{|(h,k)|} = M n^2 \dfrac{|h||k|}{\sqrt{|h|^2 + |k|^2}}$

$|(h,k)| = \sqrt{\sum h_i^2 + \sum k_j^2} = \sqrt{|h|^2 + |k|^2} \geq |h|$.

$\longrightarrow \dfrac{|f(h,k)|}{|(h,k)|} \leq M n^2 \dfrac{|h||k|}{|h|} = M n^2 |k|$

The necessary condition for $(h,k) \rightarrow 0$ is that $k \rightarrow 0$

Therefore: $\dfrac{|f(h,k)|}{|(h,k)|} \xrightarrow[(h,k)\to 0]{} 0$

b)

$f(a+x, b+y) - f(a,b) = f(a, b+y) + f(x, b+y) - f(a,b)$

$= \cancel{f(a,b)} + f(a,y) + f(x,b) + f(x,y) - \cancel{f(a,b)}$

$= f(a,y) + f(x,y) + f(x,b)$

Claimed: $Df(a,b)(x,y)$

$\Rightarrow \dfrac{|f(a+x, b+y) - f(a,b) - [f(a,y) + f(x,b)]|}{|(x,y)|} = \dfrac{|f(x,y)|}{|(x,y)|} \xrightarrow[(x,y)\to 0]{} 0$

2-12-c) It is easy to see that $\begin{cases} p: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R} \\ (x,y) \longmapsto x\cdot y \end{cases}$ is bilinear.

$\longrightarrow Dp(a,b)(x,y) = p(x,b) + p(a,y) = a\cdot y + b\cdot x.$

2-13) a) Inner product is bilinear $\longrightarrow$ by the previous question:

$$D\,IP(a,b)(x,y) = IP(a,y) + IP(x,b).$$

b) $h(t) = IP(f(t), g(t))$.

$$\to \frac{d}{dt}h(t) = D\,IP(f(t),g(t))\big(\underbrace{(f')^T}_{\left(\frac{d}{dt}f\right)^T}, \overbrace{(g')^T}^{\left(\frac{d}{dt}g\right)^T}\big)$$

proved in (a)

$$= IP(f(t),(g')^T) + IP((f')^T, g)$$

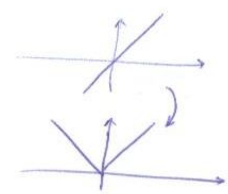
$$= \langle f(t),(g')^T\rangle + \langle (f')^T, g\rangle. \quad \blacksquare$$

c) $|f(t)| = 1 \Longrightarrow \langle f(t),f(t)\rangle = 1 \Longrightarrow \frac{d}{dt}\langle f(t),f(t)\rangle = 0$

But $\frac{d}{dt}\langle f,f\rangle = \langle f,(f')^T\rangle + \langle (f')^T,f\rangle = 2\langle f,f'\rangle = 0$

$\to \langle f,(f')^T\rangle = 0.$

d) $y = x$ is differentiable.
$y = |x|$ is not



2-15) a) I'll give the idea for the general case and prove for $n=2$.
Recall the argument in 2-12-b. It is easy to generalise it for an m-linear function. That is if
$f: \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\cdots\times\mathbb{R}^{n_m} \longrightarrow \mathbb{R}$ is m-linear,

$$Df(a_1,\dots,a_m)(x_1,\dots,x_m) = f(x_1,a_2,\dots,a_m) + f(a_1,x_2,a_3,\dots) + \cdots + f(a_1,\dots,a_{m-1},x_m).$$

2-15 - Cont'd:

In special case, determinant of a 2×2 matrix is a bilinear function. By 2-12-b, we shall get:

$$D\det \underbrace{(a_1, a_2)}_{\text{rows}}(x_1, x_2) = \det\begin{pmatrix} a_1 \\ x_2 \end{pmatrix} + \det\begin{pmatrix} a_2 \\ x_1 \end{pmatrix}$$

And by the argument for an m-linear form, it is proved for a general matrix.

Note: The formula can be re-written for the columns as well. Because det is an m-linear function of columns as well as rows.

b) Chain Rule!, as we did for 2-13-b.

$$\frac{d}{dt}\det(a_1(t), a_2(t)) = D\det(a_1, a_2)(a_1', a_2') =$$

$$\det(a_1(t), a_2'(t)) + \det(a_1'(t), a_2(t))$$

c) By Kramer's (or maybe Cramer's!) rule:

$$s_j = \frac{\det\begin{pmatrix} a^1 \\ \vdots \\ b \\ \vdots \\ a^n \end{pmatrix} \begin{matrix} \\ \\ \leftarrow j\text{-th row} \\ \\ \\ \end{matrix}}{\det\begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix}}$$

where a^k is the k-th row of matrix (a_{ij}) and $\vec{b} = (b_1, \dots, b_n)$

$\rightarrow s_j$ is the quotient of two differentiable function and by assumption the denuminator is non-zero.

$\rightarrow s_j$ is differentiable.

$$\frac{d}{dt} s_j = \frac{\frac{d}{dt}\det\begin{pmatrix} a^1 \\ b \\ a^n \end{pmatrix}\det(a_{ij}) - \det\begin{pmatrix} a^1 \\ b \\ a^n \end{pmatrix}\frac{d}{dt}\det(a_{ij})}{\left(\det\begin{pmatrix} a^1 \\ a^n \end{pmatrix}\right)^2}$$

2-20; Notation Remark: by $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$ I mean D_1 and D_2.

a) $\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} g(x) \cdot h(y) + g(x) \cdot \underbrace{\frac{\partial}{\partial x} h(y)}_{0} = (D_1 g) \cdot h(y) = \frac{d}{dx} g \cdot h(y)$.

$\frac{\partial}{\partial y} f(x,y) = 0 + g(x) \frac{d}{dy} h(y)$.

b) $\frac{\partial}{\partial x} (g(x))^{h(y)} = (h(y)) \cdot \frac{d}{dx} g(x) \cdot (g(x))^{(h(y)-1)}$

$\frac{\partial}{\partial y} f = \frac{d}{dy} h(y) \ln(|g(x)|) \cdot g(x)^{h(y)}$.

d) $D_1 f = 0$

$D_2 f = D_2 g = \frac{d}{dy} g(y)$.

c) $D_1 f = \frac{d}{dx} g(x)$

$D_2 f = 0$.

e) $D_1 f = \overset{D_1(x+y)}{1} \cdot g'(x+y)$.

$D_2 f = 1 \cdot g'(x+y)$.

Note; $g'(x+y)$ means $\left.\frac{d}{dt} g(t)\right|_{t=x+y}$.

2-21) a) $f(x,y) = \int_0^x g_1(t,0)\, dt + \int_0^y g_2(x,t)\, dt$

$D_2 f = D_2\left(\int_0^x g_1\right) + \underbrace{D_2\left(\int_0^y g_2(x,t)\, dt\right) = 0 + g_2(x,t)\big|_{t=y}}_{\text{differentiation of integral}}$

$= g_2(x,y)$.

b) Define $\tilde{f} = \int_0^x g_1(t,y)\, dt + \int_0^y g_2(x,t)\, dt$

c) let $\begin{cases} g_1(x,y) = x \\ g_2(x,y) = y \end{cases} \rightarrow \begin{cases} D_1 \tilde{f} = g_1 = x. \\ D_2 \tilde{f} = g_2 = y \end{cases}$ ($\tilde{f}$ defined in (b))

2-21-c) Cont'd;

Again for $\tilde{f}(\cdot)$ defined in "b" let $\begin{cases} g_1 = y \\ g_2 = x \end{cases} \rightarrow \begin{array}{l} D_1 f = y \\ D_2 f = x \end{array}$

2-24) $D_1 f = \qquad (x,y) \neq 0. \qquad \dfrac{4y^3x^2 - y^5 + x^4y}{(x^2+y^2)^2}$

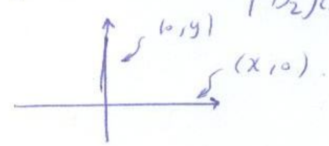
$D_2 f = \qquad (x,y) \neq 0 \qquad \dfrac{x^5 + xy^4 - 4y^2x^3}{(x^2+y^2)^2}$

substituting $(0,y)$ and $(x,0) \rightarrow D_1 f(0,y) = -y. \quad y \neq 0$

$D_2 f(x,0) = x. \quad x \neq 0$

We have to check for $(x,y) = 0$ separately.

But as $f(x,y)$ is zero on both axes, $D_1 f(0,0)$ and $D_2 f(0,0)$ both exist and equal zero. $\rightarrow \begin{cases} D_1 f(0,y) = y & \forall y \\ D_2 f(x,0) = x & \forall x \end{cases}$



(b) To find $D_{12} f$ ~~on~~ at ~~the~~ any point, we are in fact calculating $D_1 D_2$

But finding $D_1 D_2$ merely need the values of D_2 when "x" is changing. $\Rightarrow D_1 D_2 f(x,0) =$

$$\lim_{h \to 0} \frac{D_2 f(x+h, 0) - D_2 f(x,0)}{h} = \lim \frac{x+h-x}{h} = 1.$$

In a similar fashion, $D_2 D_1 f(0,y) = -1$.

which are not equal.

Note; ~~In noway~~

Noway does it contradict Thm 2-5.

(Exercise) (Why?)

HW3-6.

HW

2-25)

First derivative;

The only problematic point ~~my~~ might be $x=0$, because off this point, ~~both~~ we have got a smoot function: constant in one side and composition of two smooth functions: namely $1/x^2$ and $\exp(\cdot)$.

$$\frac{f(h)-f(0)}{h} = \begin{cases} \dfrac{\exp(-h^{-2})-0}{h} & h>0 \\[2mm] \dfrac{0-0}{h} & h<0. \end{cases}$$

$$\lim_{h\to 0^+} \frac{\exp(-h^{-2})}{h} = \lim_{t\to+\infty} \frac{\exp(-t^2)}{1/t} = 0.$$

exponential function ~~decades~~ decays faster than any polynomial.

Assume that we've proved that all derivatives up to order $k-1$ are zero at $x=0$.

Moreover, note that the derivative: $f'(x) = P_1(1/x)\exp(-1/x^2)$, $x>0$

where $P_1(\cdot)$ is a polynomial.

Inductively, if $f^{(k)}(x) = \underbrace{P_k(1/x)}_{\text{a polynom.}} \exp(-1/x^2)$; $(x>0)$.

$f^{(k)}$ ← k-th derivative

then an easy calculation shows that

$$f^{(k+1)}(x) = P_{k+1}(1/x)\exp(-1/x^2) \; ; (x>0).$$

For $x=0$;

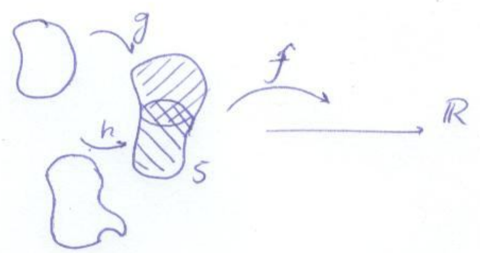
~~lim f(h)~~ $\lim_{x\to 0^+} \dfrac{\frac{d^k}{dx^k} f(x)}{x} = \lim_{t\to+\infty} \left[t\, P_k(t) \exp(-t^2)\right] = 0$

$\to f^{(k)}(0) = 0 \quad (\forall k)$

again because $\exp(\cdot)$ dies faster.

HW3-7

2-27; It suffices to show that g & h cover the domain of "f" completely;.



The domain of "f" is the set $S^* = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$.

$\rightarrow$ if $z \geq 0 \Longrightarrow z = \sqrt{1-x^2-y^2}$ $\quad x^2+y^2 \leq 1$

$z \leq 0 \Longrightarrow z = -\sqrt{1-x^2-y^2}$ $\quad x^2+y^2 \leq 1$.

$z \geq 0$ is covered by 'g' and $z \leq 0$ by 'h'.

Note; In fact g covers the upper hemisphere of S^* and h the lower one. The equator, $z=0$, is covered by both.