MAT322 - HW12

4.1; a) By definition : $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(e_{i_1}, \ldots, e_{i_k}) = \frac{k!}{1! \cdots 1!} \sum \operatorname{sgn}(\sigma)$

$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(e_{i_{\sigma(1)}}, \ldots, e_{i_{\sigma(k)}})$ $(*)$

But $\varphi_i(e_j) = \delta_{ij}$ $\rightarrow$ $(*) = 1 + 0 + \cdots + 0 = 1$

b) $v_k = \sum v_k^i e_i$ $\rightarrow$

$(*)$ $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_1, \ldots, v_k) = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(\sum v_1^i e_i, \ldots, \sum v_k^i e_i)$

But, again as $\varphi_i(e_j) = \delta_{ij}$, if $I = \{i_1, \ldots, i_k\}$.

$(*) = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(\sum_{i \in I} v_1^i e_i, \ldots, \sum_{i \in I} v_k^i e_i)$

which is

$$\det \begin{bmatrix} v_1^{i_1} & \cdots & v_1^{i_k} \\ \vdots & & \\ v_k^{i_1} & \cdots & v_k^{i_k} \end{bmatrix}$$

4.2.

If $\dim V = n \Rightarrow \dim \Lambda^n = \binom{n}{n} = 1$. That is f^*, as a linear transformation on Λ^n can only be a multiplication. let $\omega \in \Lambda^n$, and let 'c' be as defined in the book. $\rightarrow f^*(\omega) = c\omega$

But $f^*(\omega)(e_1, \dots, e_n)$

$$= \omega(f(e_1), \dots, f(e_n)) =$$

$$\omega\left(\sum f_{i1} e_i , \sum f_{i2} e_i, \dots , \sum f_{i3} e_i \right) =$$

By 4-6: $\det(f_{ij})\, \omega(e_1, \dots, e_n)$

$$\Rightarrow c = \det(f_{ij})$$

4-3; Following the hint; let $w_i = \sum_k a_{ik} v_k \rightarrow g_{ij} = T(w_i, w_j) =$.

$$T\left(\sum_k a_{ik} v_k , \sum_k a_{jk} v_k\right) = \sum_m a_{im} a_{jm}$$

by expanding and noting that $T(v_k, v_l) = \delta_{kl}$

$$\Rightarrow (g_{ij}) = (a_{ij})(a_{ij})^T \Rightarrow |\omega(w_1, \dots, w_n)|^2 = |\det(a_{ij})|^2 = \det(g_{ij})$$

MAT322 - HW12;

4.9; a) As determinant is alternating, $v \times v = 0$

To show that $e_1 \times e_2 = e_3$, it suffices to prove that for

$w = e_1, e_2, e_3$: $\langle w, e_3 \rangle = \det\begin{pmatrix} e_1 \\ e_2 \\ w \end{pmatrix}$, which is obvious.

b) let $\langle w, e_i \rangle = \det\begin{pmatrix} v \\ w \\ e_i \end{pmatrix}$

and $w = \langle w, e_1 \rangle e_1 + \langle w, e_2 \rangle e_2 + \langle w, e_3 \rangle e_3$.

or, we can use 'a' and note that

$$\begin{cases} w = w^1 e_1 + w^2 e_2 + w^3 e_3 \\ v = v^1 e_1 + w^2 e_2 + w^3 e_3 \end{cases}$$

$w \times v = (w^1 e_1 + w^2 e_2 + w^3 e_3) \times (v^1 e_1 + v^2 e_2 + v^3 e_3) =$

expanding and the facts of part 'a' gives the result.

c) $\|v \times w\|^2 = \|v\|^2 \|u\|^2 - (u \cdot v)^2 = \|v\|^2 \|u\|^2 (1 - \cos^2\theta)$

verify!

$= \|v\|^2 \|u\|^2 \sin^2\theta$

$\langle v \times w, v \rangle = \det\begin{pmatrix} v \\ w \\ v \end{pmatrix} = 0 = \det\begin{pmatrix} v \\ w \\ w \end{pmatrix} = \langle w, v \times w \rangle$.

d) $\langle v, w \times z \rangle = \det\begin{pmatrix} w \\ z \\ v \end{pmatrix} = \det\begin{pmatrix} z \\ v \\ w \end{pmatrix} = \langle w, z \times v \rangle = \langle z, v \times w \rangle$.

similarly

Check that for $i, j = 1, 2, 3$,

$v \times (e_i \times e_j) = \langle v, e_j \rangle e_i - \langle v, e_i \rangle e_j$ holds. By

linearity, $v \times (w \times z) = \langle v, z \rangle w - \langle v, w \rangle z$.

$(v \times w) \times z = - z \times (v \times w) = - \langle w, z \rangle v - (-\langle v, z \rangle w)$.

MAT322 - HW12

4.9 - e). This is what we used in 'c'!

We used expansion. (supposedly!)

4 - 10; If $\{w_1, \dots, w_{n-1}\}$ is not a set of linearly independent sets, $\Rightarrow \begin{cases} \det g_{ij} = 0 \\ w_1 \times \cdots \times w_{n-1} = 0 \end{cases}$

So, let it be linearly independent. Consider the $(n-1)$-dim'l subspace $W = \{w_1, \dots, w_{n-1}\}$. $\Rightarrow \dim W^{\perp} = 1$ Define

$f = \dfrac{w_1 \times \cdots \times w_{n-1}}{|w_1 \times \cdots \times w_{n-1}|} \Rightarrow \forall\, \tilde{w}_1, \dots, \tilde{w}_{n-1} \in W \quad \exists c : \text{s.t.}:$

$cf = \tilde{w}_1 \times \cdots \times \tilde{w}_{n-1}$. Define $\psi(\tilde{w}_1, \dots, \tilde{w}_{n-1}) = c$.

ψ is an $(n-1)$-form on W.

ψ is a volume form on $W \rightarrow$ By 4.3

$|\psi(w_1, \dots, w_{n-1})| = |c| = |w_1 \times \cdots \times w_{n-1}| = \sqrt{\det g_{ij}}$

HW12 - 4.