MAT 322 - HW 11

3-35; let the rectangle be $R=[a_1,b_1]\times\cdots\times[a_n,b_n]$. (It is easy to see that open or closed rectangles ~~have~~ make no difference here).

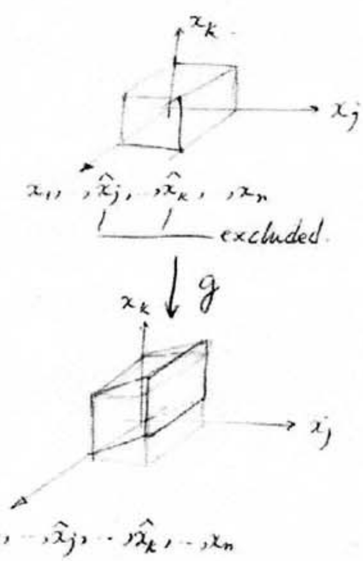
$$\begin{cases} g(e_i)=e_i & i\neq j \\ g(e_j)=\alpha e_j \end{cases} \Rightarrow g(R)=[a_1,b_1]\times\cdots\times[\alpha a_j,\alpha b_j]\times\cdots\times[a_n,b_n]$$

$$\det g=\det\begin{bmatrix} 1 & & 0 \\ & \alpha & \\ 0 & & 1 \end{bmatrix}=\alpha$$

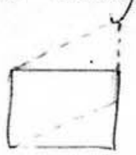
Done!

$$\begin{cases} g(e_i)=e_i & i\neq j \\ g(e_j)=e_j+e_k \end{cases}$$

$$\det g=\det \begin{array}{c} \quad\quad j \quad\quad k \\ \begin{bmatrix} 1 & & & \\ & 1 0 0 \cdots 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \end{array}=1$$



By the same trick in elementary mathematics, when we proved that the area of parallelogram and rectangle are equal:



you can see that the area of parallelpiped and rectangle are equal.

$\triangle$ S : You have to be more rigorous than this. in the proof!

$$\begin{cases} g(e_k)=e_k & k\neq i,j \\ g(e_i)=e_j \\ g(e_j)=e_i \end{cases}$$

$$g(R)=[a_1,b_1]\times\cdots\times\underbrace{[a_j,b_j]}_{i\text{-th}}\times\cdots\times\underbrace{[a_i,b_i]}_{j\text{-th}}\times\cdots\times[a_n,b_n]$$

$$v(g(R))=v(R)$$

$$\det g=\det\begin{bmatrix} 1 & & & & \\ 0\cdots 1 & 0 & & & 0 \\ & 1 & \ddots & & 1 \\ & & & 0 & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}=1$$

HW 11 # 1)

b). We know from linear algebra, that for any invertible matrix [g]

$$g = g_1 \circ \cdots \circ g_m \circ I = g_1 \circ \cdots \circ g_m$$

for opropriate g_i's. $\Rightarrow |\det g| = \prod_{i=1}^{m} \det |g_i|$

If g is not invertible, $g(R)$ is a ~~non~~ degenerate rectangle of volume '0'.

3-41: a) injectivity is checked easily.

$$\frac{\partial f(r,\theta)}{\partial (r,\theta)} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \rightarrow \det Df = r > 0$$

b) You can invert Df, if you need the inverse in ter. of (r,θ), or see, directly, that: (above "in ter.", with a caret: derivative.)

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-1}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

c) By change of variables, and $\frac{\partial(x,y)}{\partial(r,\theta)} = r$

$$\int_C h = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} r g \, d\theta dr$$

let $r_1 \to 0$,

$\int_{B_r} h = \int_0^r\int_0^{2\pi} rg \, d\theta dr$. Note that polar coordina is not defined at the origin, nevertheless, as h is integrable on B_r, the value of one point does not change anything.

c) $\int_{B_r} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi}\int_0^r re^{-r^2} dr d\theta = -\pi e^{-r^2}\Big|_0^r = \pi(1-e^{-r^2})$

d) $\int_C e^{-(x^2+y^2)} dx dy = \int_{-r}^r e^{-x^2} \int_{-r}^r e^{-y^2} dx dy = \left(\int_{-r}^r e^{-x^2}dx\right)\left(\int_{-r}^r e^{-y^2}dy\right)$

$$= \left(\int_{-r}^r e^{-x^2}dx\right)^2.$$

e: 1+d) $\int_{C_{r/2}} e^{-(x^2+y^2)}dx dy \le \int_{B_r} e^{-(x^2+y^2)}dx dy \le \int_{C_r} e^{-(x^2+y^2)}dx dy$.

not?

So $\lim \int_{C_r} \cdots = \lim \int_{B_r} \cdots \Rightarrow \pi(1-e^{-r^2}) \xrightarrow[r\to\infty]{} \pi$

$$\Rightarrow \boxed{\left(\int_{-r}^{r} e^{-x^2}\right)^2 = \pi}$$

Splendid! Isn`t it?

3-40: Revisited:

I shall give two solutions. The first one is the very proof given in the previous post, clarified by a change of notation and some more details.

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1) Assume $g(0)=0$. We procede inductively. Assume we've got a function $\Gamma_j(x^1, \ldots, x^n) = (x^1, \ldots, x^{j-1}, p^j, \ldots, p^n)$.

Note that $g(\cdot)$ is of the form $\Gamma_0 = (g^1, \ldots, g^n)$.

$$\rightarrow D\Gamma_j = \left(\begin{array}{c|c} I_{(j-1)\times(j-1)} & 0 \\ \hline \multicolumn{2}{c}{Dp^j} \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{Dp^n} \end{array} \right).$$

Moreover, assume $\Gamma_j(0)=0$, and $D\Gamma_j(0)$ is non-singular. Then,

$$(D\Gamma_j(0))(\hat{e}_j) = j\text{-th column of } D\Gamma_j(0) \neq 0$$

by $D\Gamma_j$ being non-singular.

Note that j-th column of $D\Gamma_j(0) = \left.\begin{bmatrix} 0 \\ \vdots \\ 0 \\ D_j p^j \\ \vdots \\ D_j p^n \end{bmatrix}\right\} j-1 \neq 0.$

$\Rightarrow$ for some "k": $j \le k \le n$: $D_j p^k \neq 0$.

Define $g_j(x^1, \ldots, x^n) = (x^1, \ldots, x^j, p^k, x^{j+1}, \ldots, x^n)$

$$\rightarrow Dg_j(0) = \begin{pmatrix} 1 & & & 0 \\ & \ddots 1 & & \\ & & D_j p^k & \\ 0 & & & \ddots \\ & & & 1 \end{pmatrix} \rightarrow \text{By } D_j p^k \neq 0 \rightarrow$$

$\det(Dg_j(0)) \neq 0 \Rightarrow g_j(\cdot)$ is, by Inverse function thm, invertible (locally).

3-40 - Con'd:

We constructed g_j by Γ_j. Let 'T_j' be a linear transformation which exchanges 'k' and 'j'.

$\Rightarrow$ And $(g_j)^{-1} = (x^1, \ldots, x^j, q^k, x^{j+1}, \ldots, x^n)$.

[It is easy to verify that $(g_j)^{-1}$ has to be of this form].

k-th place

$\rightarrow \Gamma_j \circ g_j^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^{j-1}, p^j, \ldots, \underline{\underline{x^j}}, \ldots p^n)$

$\Rightarrow T_j \circ \Gamma_j \circ g_j^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^{j-1}, x^j, p^{j+1}, \ldots, p^j, \ldots, p^n)$

k-th place

Now, $T_j \circ \Gamma_j \circ g_j^{-1}$ is of the form admissible for Γ_{j+1}. $\Rightarrow$

~~Note that $g = \Gamma_0$~~

Define $\Gamma_{j+1} := T_j \circ \Gamma_j \circ g_j^{-1} \Rightarrow T_j^{-1} \circ \Gamma_{j+1} \circ g_j = \Gamma_j$

$\rightarrow$ Note that $g = \Gamma_0 \Rightarrow g = \Gamma_0 = T_0^{-1} \circ \Gamma_1 \circ g_0 =$

$T_0^{-1} \circ (T_1^{-1} \circ \Gamma_2 \circ g_1) \circ g_0 = T_0^{-1} \circ T_1^{-1} \circ T_2^{-1} \circ \Gamma_3 \circ g_2 \circ g_1 \circ g_0 = \cdots$

$\cdots = \underbrace{T_0^{-1} \circ \cdots \circ T_{n-1}^{-1}}_{T} \circ \underbrace{\Gamma_n}_{g_n} \circ g_{n-1} \circ \cdots g_1 \circ \underbrace{g_0}_{id}$

which is the desired form.

HW 12 - App - II

Method II: This is gotten from ~~your~~ some of students' homework.

Assume $Dg(0) = I$, if not, replace $g(\cdot)$ by $(Dg(0))^{-1} \circ g$

If $g = (h^1, \dots, h^n)$, define:

$$\cancel{g_1} \quad g_1(x^1, \dots, x^n) = (h^1, x^2, \dots, x^n)$$

$$Dg_1(0) = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix} \quad Dh^1 = (1, 0, \dots, 0) \text{ by assumption}$$

$\Rightarrow g_1$ is locally invertible. $\Rightarrow$

$$(g_1)^{-1} = (\phi^1, x^2, \dots, x^n).$$

Define $I_1 := g \circ (g_1)^{-1} \Rightarrow$

$$I_1(x^1, \dots, x^n) = (x^1, h^2, \dots, h^n).$$

~~neglecting the first~~ Apply the same to I_1 to get $I_2 = (x^1, x^2, \tilde{h}^3, \dots, \tilde{h}^n)$ such that $I_2 = I_1 \circ (g_2)^{-1}$ and so on.

$$g = I_1 \circ g_1 = I_2 \circ g_2 \circ g_1 = \cdots = g_n \circ \cdots \circ g_1$$