HW10 – MAT322.

3-32; Note that this exercise claims that under 'certain conditions' you can differentiate with respect to the variable which is not involved in integration under the integral sign.

We just need to assume that $D_2 f$ is 'integrable'. Then, by fundamental thm of calculus in one variable:

$$f(x,y) = \int_c^y D_2 f(x,y')\,dy' + f(x,c)$$

$$\to \frac{d}{dy}F(y) = \frac{d}{dy}\int_a^b f(x,y)\,dx = \frac{d}{dy}\int_a^b \Big[\int_c^y D_2 f(x,y')\,dy' + \underbrace{f(x,c)}_{\text{disappears in differentiation}}\Big]dx$$

by Fubini's: $\displaystyle \frac{d}{dy}\int_c^y \Big(\int_a^b D_2 f(x,y')\,dx\Big)dy' =$

Fundamental Thm of calculus: $\displaystyle \int_a^b D_2 f(x,y')\,dx$

3-33; a) $D_1 F(x,y) = f(x,y)$.

$D_2 F(x,y) = \int_a^x D_2 f(t,y)\,dt$ as the conditions of 3-32 are satisfied.

b) Let's derive a more general rule;

Define $H(x,y) = \int_a^{g(x)} f(t,p(y))\,dt$

$\to \frac{\partial H}{\partial x} = g'(x)\cdot f(g(x),p(y))$ chain rule + Fundamental Thm of calculus. (part 'a')

$$\frac{\partial H}{\partial y} = p'(y)\int_a^{g(x)} D_2 f(t, p(y))\, dt \quad \text{chain rule + part 'a'.}$$

$\rightarrow$ Note that $G(x) = H(x, x)$, when $p(y) = y$. $\Rightarrow p'(y) = 1$.

$$\Rightarrow \frac{d}{dx} G(x) = D_1 H + D_2 H = g'(x) f(x, p(y)) + \int_a^{g(x)} D_2 f(t, p(y))\, dt$$

3-34; $D_1 f(x,y) = D_1 \int_0^x g_1(t,0)\, dt + \int_0^y D_1 g_2(x,t)\, dt =$

$$= g_1(t,0) + \int_0^y D_2\, g_1(x,t)\, dt$$

$$= g_1(t,0) + (g_1(x,y) - g_1(x,0)) = g_1(x,y). \quad \blacksquare$$

3-37; a) First of all, note that if f is integrable $\rightarrow$ For the partition of unity $\{\phi_n\}$: $\sum \phi_n |f| = \sum_i \phi_n f < \infty$

Moreover: $\underbrace{\int_\varepsilon^{1-\varepsilon} f}_{g(\varepsilon)} \leq \sum_{n \in \circ}^{\infty} \phi_n f < \infty \Rightarrow$ If 'f' is integrable,

$\lim_{\varepsilon \to 0} g(\varepsilon)$ exists (note that $g(\varepsilon)$ is monotone.

On the other hand:

(b) We know that $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{(-1)^i}{i} = \log 2$ by this fact,

$\lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} f = \log 2$

But $\int_{(0,1)} |f| \geq \sum_n \int_{A_n} |f| \geq \sum_n \left|\int_{A_n} f\right| = \sum_n 1/n = \infty$

$\rightarrow f$ is not integrable. : $\int_{(0,1)} |f| = \infty$.