## Section 3.6

**#4)** (a) Let $M > 0$. Let $K \in \mathbb{N}$ be greater than $M^2$. Then for all $n \geq K$, $\sqrt{n} > M$. Hence, $\sqrt{n}$ is properly divergent to $+\infty$.

(b) $\sqrt{n} < \sqrt{n+1}$, so by the Comparison Theorem, $\sqrt{n+1}$ tends to $+\infty$.

(c) $\sqrt{n}$ is the 1-tail of $\sqrt{n-1}$, so $\sqrt{n-1}$ tends to $+\infty$ since its tail does.

(d) $\lim((n/\sqrt{n+1})/\sqrt{n+1}) = \lim(n/(n+1)) = 1$, so by (b) and the Limit Comparison Test, $n/\sqrt{n+1}$ tends to $+\infty$.

**#5)** No. In Section 3.4, subsequences $(x_{n_k})$ and $(y_{n_j})$ of $(\sin n)$ were constructed such that $x_{n_k} \geq 1/2$ and $y_{n_j} \leq -1/2$. Thus, $n_k x_{n_k}$ tends to $+\infty$ and $n_j y_{n_j}$ tends to $-\infty$. Therefore, $(n \sin n)$ has two subsequences tending toward different infinities.

**#6)** $(x_n)$ is properly divergent, so $(|x_n|)$ tends to $+\infty$. There exists $K$ such that $n \geq K$ implies $|x_n| > 0$. $(x_n y_n)$ converges, and so is bounded. Let $M$ be an upper bound for $|x_n y_n|$. Then for $n \geq K$, $0 \leq |y_n| \leq \frac{M}{|x_n|}$. By the Squeeze Theorem, $(|y_n|) \to 0$ and hence $(y_n) \to 0$.

**#7)** (a) Let $K$ be such that for $n \geq K$, $0 < x_n/y_n < 1$. Thus, for the $K$-tail, $x_n < y_n$. By the Comparison Theorem, $(y_n) \to +\infty$.

(b) Let $M$ be an upper bound for $y_n$. Then $0 < x_n \leq M\frac{x_n}{y_n}$, and so be the Squeeze Theorem, $(x_n) \to 0$.

**#8)** (a) $\sqrt{n^2+2} > n$, so the sequence properly diverges by the Comparison Test.

(b) $0 < \sqrt{n}/(n^2+1) < 1/n$, so the sequence converges to 0.

(c) $\sqrt{n^2+1}/\sqrt{n} > \sqrt{n}$, so the sequence properly diverges by the Comparison test.

(d) $(\sin \sqrt{n})$ is bounded, and so does not properly diverge. It does diverge, however, seeing as how it contains a divergent subsequence $(\sin k)$ (take $n_k = k^2$).

**#9)** (a) Let $K$ be such that for $n \geq K$, $x_n/y_n > 1$. Thus for the $K$-tail, $x_n > y_n$. By the Comparison Theorem, $(x_n) \to +\infty$.

(b) Let $M$ be an upper bound for $x_n$. Then $x_n/y_n \leq M/y_n$, and by the Comparison Theorem, $(M/y_n) \to \infty$, and hence $1/y_n \to \infty$. It follows that $y_n \to 0$.

**#10)** Since $L > 0$, there exists $K$ such that for $n \geq K$, $a_n/n > 0$. So for the $K$-tail, $a_n > 0$. Since $n$ is properly divergent, we can apply the Limit Comparison Test to the $K$-tail to achieve the result: $(a_n) \to +\infty$.

## Section 3.7

**#2)** Let $\sum x_n$ be a convergent series, and let $\sum y_n$ be a series such that $x_n = y_n$ for all but finitely many $n$. Then there exists $K$ such that $x_n = y_n$ for all $n \geq K$. Let $\epsilon > 0$. There exists $M \geq K$ such that for all $m > n \geq M$, $\left|\sum_{k=n+1}^{m} x_n\right| < \epsilon$. But this sum is equal to $\left|\sum_{k=n+1}^{m} y_n\right|$. And so the series $\sum y_n$ also satisfies the Cauchy Criterion for series, whence converges.

**#4)** Let $(s_n)$ and $(t_n)$ be the sequences of partial sums associated with $\sum x_k$ and $\sum y_k$, respectively. By assumption, $(s_n)$ and $(t_n)$ converge, and so $(s_n + t_n)$ converges. But $s_n + t_n = (x_1 + \cdots + x_n) + (y_1 + \cdots + y_n) = (x_1 + y_1) + \cdots + (x_n + y_n)$ is the sequence of partial sums associated with $\sum(x_k + y_k)$, so $\sum(x_k + y_k)$ converges.

**#5)** No. Suppose $\sum x_n$ and $\sum(x_n + y_n)$ are convergent. Then $\sum -x_n$ is convergent, and by #4, $\sum y_n = \sum(x_n + y_n - x_n)$ must be convergent.

**#6b)** First, $\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ for all $n$, so $\sum \frac{|\cos n|}{n^2}$ converges. Thus, it is Cauchy. Let $\epsilon > 0$. There exists $K$ such that for all $m > n \geq K$, $\sum_{k=n+1}^{m} \frac{|\cos n|}{n^2} < \epsilon$. By the Triangle Inequality, $\left|\sum_{k=n+1}^{m} \frac{\cos n}{n^2}\right| \leq \sum k = n + 1^m \frac{|\cos n|}{n^2} < \epsilon$. So $\sum \frac{\cos n}{n^2}$ is Cauchy, and hence convergent.

**#7)** Let $(s_n)$ be the sequence of partial sums associated with the series. Then

$$s_{2n} = (-1 + \frac{1}{\sqrt{2}}) + \cdots + (\frac{-1}{\sqrt{2n-1}} + \frac{1}{\sqrt{2n}})$$

is a decreasing sequence, with upper bound 0. Also,

$$s_{2n+1} = -1 + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}) + \cdots + (\frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}})$$

is an increasing sequence with lower bound -1. Therefore, $-1 \leq s_{2n+1} = s_{2n} - \frac{1}{\sqrt{2n+1}} < s_{2n} \leq 0$. By the Monotone Convergence Theorem, $s_{2n+1}$ and $s_{2n}$ both converge, and since $\frac{1}{\sqrt{2n+1}} \to 0$, the limit is the same. It follows that $(s_n)$ converges, and hence the series converges.

**#8)** Yes, $\sum a_n$ converges, so $\lim(a_n) = 0$. So there exists $K$ such that for $n \geq K$, $0 < a_n < 1$. Therefore, for $n \geq K$, $0 < a_n^2 < a_n$. By the Comparison Theorem, $\sum a_n^2$ converges.

**#9)** No. Take $a_n = 1/n^2$ so that $\sqrt{a_n} = 1/n$.

**#10)** Yes. By the arithmetic-geometric mean inequality, $\sqrt{a_n a_{n+1}} \leq \frac{1}{2}(a_n + a_{n+1})$. $\sum a_n$ converges, to $\sum a_{n+1}$ also converges. By problem 4, $\sum(a_n + a_{n+1})$ converges. By the Comparison Theorem, $\sum \sqrt{a_n a_{n+1}}$ converges.