

## Section 3.4

- #4) (a)  $x_n = 1 - (-1)^n + 1/n$ . We take the subsequence of the even-indexed terms:  $x_{n_k} = x_{2k} = 1 - (-1)^{2k} + 1/(2k) = 2 + 1/(2k)$ . So  $(x_{2k})$  converges to 2. Next we take the subsequence of odd-indexed terms:  $x_{2k-1} = 1 - (-1)^{2k-1} + 1/(2k-1) = 1/(2k-1)$ . So  $(x_{2k-1})$  converges to 0. Since there are two subsequences converging to different limits, the entire sequence diverges.
- (b)  $x_n = \sin(n\pi/4)$ . We first take the subsequence of multiples of 8:  $x_{8k} = \sin(8k\pi/4) = \sin(2k\pi) = 0$ . So the subsequence  $(x_{8k})$  converges to 0. Now we take the subsequence where  $n_k = 8k + 2$ :  $x_{8k+2} = \sin((8k+2)\pi/4) = \sin(2\pi k + \pi/2) = \sin(\pi/2) = 1$ . So  $(x_{8k+2})$  converges to 1. Since there are two subsequences converging to different limits, the entire sequence diverges.

#6) (a)

$$\begin{aligned} x_{n+1} < x_n &\iff (n+1)^{1/(n+1)} < n^{1/n} \iff (n+1)^n < n^{n+1} \\ &\iff \frac{(n+1)^n}{n^n} < n \iff \left(1 + \frac{1}{n}\right)^n < n \end{aligned}$$

It was shown in Example 3.3.6 that  $(1+1/n)^n < 3$ . Hence, for  $n \geq 3$ , we have  $(1+1/n)^n < 3 \leq n$ , and therefore  $x_{n+1} < x_n$  for all  $n \geq 3$ . Thus, the sequence is ultimately decreasing (its 3-tail is decreasing) and since it is bounded below by 1,  $x := \lim(x_n)$  exists. Also  $x \geq 1$ .

- (b)  $(x_{2n})$  also converges to  $x$ , and  $x_{2n} = (2n)^{1/(2n)} = \sqrt{2^{1/n}} \sqrt{n^{1/n}} = \sqrt{2^{1/n}} \sqrt{x_n}$ . Taking the limit of both sides and recalling that  $2^{1/n}$  converges to 1, we arrive at  $x = \sqrt{x}$ . Since  $x \geq 1$ , the only viable solution is  $x = 1$ .
- #7) (a)  $\left((1+1/n^2)^{n^2}\right)$  is a subsequence of  $\left((1+1/N)^N\right)$ , which we know converges to  $e$ . Therefore, the sequence in question also converges to  $e$ .
- (b)  $\left((1+1/2n)^{2n}\right)$  is a subsequence of  $\left((1+1/N)^N\right)$ , which we know converges to  $e$ . Therefore, the subsequence also converges to  $e$ . Hence,  $\left((1+1/2n)^n\right) = \left(\sqrt{(1+1/2n)^{2n}}\right) \rightarrow \sqrt{e}$ .
- (c)  $\left((1+1/n^2)^{2n^2}\right) = \left(\left((1+1/n^2)^{n^2}\right)^2\right)$ . So the sequence in question converges to  $e^2$  by part (a) and the algebraic properties of limits.
- (d) There is no way to manipulate  $(1+2/n)^n$  to get a subsequence of  $(1+1/N)^N$ . By problem 5 (proven below), it suffices to show that the even-indexed and odd-indexed subsequences  $(1+2/(2k))^{2k}$  and  $(1+2/(2k-1))^{2k-1}$  converge to the same thing. Even-indexed sequence:  $(1+2/(2k))^{2k} = (1+1/k)^k$ , which converges to  $e^2$ . Odd-indexed sequence: note that

$$(1+2/(2k))^{2k-1} < (1+2/(2k-1))^{2k-1} < (1+2/(2k-2))^{2k-1}.$$

Also,  $(1+2/(2k))^{2k-1} = \frac{(1+1/k)^{2k}}{(1+1/k)} \rightarrow e^2$ . Similarly, the upper bound also converges to  $e^2$ . Therefore, by the Squeeze Theorem, the odd-indexed subsequence converges to  $e^2$  as well.

Proof of Problem 5: Notation as in problem 5, If  $Z$  is convergent, then obviously  $X$  and  $Y$  both also converge to  $\lim Z$  since they are subsequences of  $Z$ . Conversely, assume now that  $X$  and  $Y$  converge and  $\lim X = \lim Y =: z$ . Let  $\epsilon > 0$ . Let  $K_x$  be large enough so that for  $n \geq K_x$ ,  $|x_n - z| < \epsilon$ . Let  $K_y$  be large enough so that for  $n \geq K_y$ ,  $|y_n - z| < \epsilon$ . Let  $K = \max\{2K_x - 1, 2K_y\}$ . Let  $n \geq K$ . If  $n$  is even, then  $n = 2m$  for some  $m$ , and since  $n \geq 2K_y$ , we have  $m \geq K_y$  and so  $|z_n - z| = |z_{2m} - z| = |y_m - z| < \epsilon$ . If  $n$  is odd, then  $n = 2m - 1$  for some  $m$ , and since  $n \geq 2K_x - 1$ , we have  $m \geq K_x$  and so  $|z_n - z| = |z_{2m-1} - z| = |x_m - z| < \epsilon$ .  $\square$

- #8) (a)  $\left((3n)^{1/2n}\right) = \left[\left((3n)^{1/3n}\right)^{3/2}\right]$ . Since  $(3n)^{1/3n}$  is a subsequence of  $N^{1/N}$ , which converges to one, we have that it converges to 1. Hence, the sequence in question converges to  $1^{3/2} \rightarrow 1$ .
- (b) Using the same trick as in the last part, we see that the sequence  $(1+1/2n)^{3n}$  converges to  $e^{3/2}$ .

#9) Suppose  $X$  does not converge to 0. Then there exists  $\epsilon_0 > 0$  such that for all  $K \in \mathbb{N}$ , there exists  $n > K$  such that  $|x_n| > \epsilon_0$ . Fix such an  $\epsilon_0$  and let  $n_1$  be such that  $|x_{n_1}| > \epsilon$ . Assuming  $n_k$  has been defined, by the definition of  $\epsilon_0$ , we can find  $n_{k+1} > n_k$  such that  $|x_{n_{k+1}}| > \epsilon_0$  (Take  $n_k$  to be  $K$  in the definition of  $\epsilon_0$ ). In this way, we recursively define a subsequence  $(x_{n_k})$  of  $X$ . By assumption this subsequence has a subsequence  $(x_{n_{k_j}})$  that converges to 0. For ease of notation, let  $y_j = x_{n_{k_j}}$ . Then  $y_j$  converges to 0, so in particular, there exists a  $J$  such that  $|y_j| < \epsilon_0$ . But  $(y_j)$  is a subsequence of  $(x_{n_k})$  and so  $|y_j| > \epsilon_0$  for all  $j$ . This gives us a contradiction.

#14) Since  $s = \sup\{x_n\}$ , let  $n_1$  be such that  $s - 1 < x_{n_1} \leq s$ . Moreover, since  $s$  is not a value attained in the sequence, we must have  $s - 1 < x_{n_1} < s$ . Assume  $n_k$  has been defined such that  $s - 1/k < x_{n_k} < s$ . Let  $M_k = \max\{s - \frac{1}{k+1}, x_1, x_2, \dots, x_{n_k-1}, x_{n_k}\}$ . Since  $s$  is not attained by the sequence,  $M_k < s$ . Hence we can find  $n_{k+1}$  such that  $M < x_{n_{k+1}} \leq s$ . Again, since  $s$  is not attained by this sequence, we actually have  $M < x_{n_{k+1}} < s$ . Also,  $s - \frac{1}{k+1} \leq M < x_{n_{k+1}}$ . Finally, for all  $n \leq n_k$ , we have that  $x_n \leq M$  by definition of  $M$ . It follows that  $n_{k+1} > n_k$ . Thus, we recursively get a well-defined subsequence  $(x_{n_k})$  such that  $s - 1/k < x_{n_k} < s$  for all  $k$ . By the Squeeze Theorem, it follows that  $(x_{n_k})$  converges to  $s$ .

#15) Let  $(I_n)$  be a nested sequence of nonempty closed bounded intervals. Let  $x_n \in I_n$  for all  $n$ . Then  $x_n \in I_1$  since  $I_n \subseteq I_1$  for all  $n$ , and since  $I_1$  is bounded,  $(x_n)$  is a bounded sequence. Thus, by the Bolzano-Weierstrass Theorem,  $(x_n)$  has a subsequence  $(x_{n_k})$  that converges to some element  $x$ . We need to show that  $x \in \bigcap_{n=1}^{\infty} I_n$ . Fix arbitrary  $K \in \mathbb{N}$ . We will show that  $x \in I_K$ . First note that  $n_K \geq K$ . Indeed,  $n_1 \geq 1$ , and if  $n_j \geq j$  for some  $j$ , then  $n_{j+1} \geq n_j + 1 \geq j + 1$  and the result follows by induction. Therefore,  $I_{n_K} \subseteq I_K$ . Therefore, for all  $k \geq K$ ,  $n_k \geq n_K \geq K$  and so  $I_{n_k} \subseteq I_K$ . In particular, we have  $x_{n_k} \in I_K$  for all  $k \geq K$ . If  $I_K = [a_K, b_K]$ , then we have that  $a_K \leq x_{n_k} \leq b_K$  for all  $k \geq K$ . By Theorem 3.2.6, this implies that  $a_K \leq x \leq b_K$  and  $x \in I_K$  as desired. Since  $K$  was arbitrary, we get  $x \in I_K$  for all  $K$ , and so  $x$  is in the intersection.

## Section 3.5

#1) The sequence  $((-1)^n)$  is bounded (by  $-1$  and  $1$ ), and not Cauchy (since it is not convergent).

#2a)  $(\frac{n+1}{n}) = (1 + 1/n)$ . Let  $\epsilon > 0$ . Let  $K \in \mathbb{N}$  be large enough so that  $1/K < \epsilon/2$ . Then for  $n, m \geq K$ , we have  $|(1 + 1/n) - (1 + 1/m)| = |1/n - 1/m| \leq 1/n + 1/m \leq 1/K + 1/K < \epsilon/2 + \epsilon/2 = \epsilon$ .

#3b) Let  $\epsilon = 1$ . We will show that for all  $K \in \mathbb{N}$ , there exist  $n, m \geq K$  such that  $|x_n - x_m| \geq 1$ . Fix arbitrary  $K \in \mathbb{N}$ . Let  $m$  be an odd number greater than  $K$  and  $n = m + 1$ . Then  $|x_n - x_m| = |(m+1) + \frac{(-1)^{m+1}}{m+1} - m - \frac{(-1)^m}{m}| = |1 + \frac{1}{m+1} + \frac{1}{m}| > 1$  as desired.

#6) Fix  $p \in \mathbb{N}$ . Let  $s_n = \sum_{k=1}^n \frac{1}{k}$ . Let  $p \in \mathbb{N}$ . First, the sequence  $(s_n)$  diverges (and so is not Cauchy) by 3.5.6(c) or 3.3.3(b). For all  $n$ ,  $|s_{n+p} - s_n| = \sum_{k=1}^p \frac{1}{n+k} \leq \sum_{k=1}^p \frac{1}{n} = \frac{p}{n}$ , which converges to 0. Hence,  $0 < |s_{n+p} - s_n| \leq \frac{p}{n}$ , and by the Squeeze Theorem,  $\lim |s_{n+p} - s_n| = 0$ .

#7) Let  $\epsilon = 1$ . Since  $(x_n)$  is Cauchy, there exists  $K \in \mathbb{N}$  such that for all  $n, m \geq K$ ,  $|x_n - x_m| < 1$ . In particular,  $|x_n - x_K| < 1$  for all  $n \geq K$ . It follows that for all  $n \geq K$ ,  $x_n \in (x_K - 1, x_K + 1)$ . Since  $x_K$  is an integer, it is the only integer in  $(x_K - 1, x_K + 1)$ . Therefore, since  $x_n$  is an integer, we must have  $x_n = x_K$  for all  $n \geq K$ . Therefore, the  $K$ -tail of the sequence is constant. Therefore, the sequence is ultimately constant.

#9) Let  $m, n \in \mathbb{N}$ . Without loss of generality, we may assume  $m \geq n$ . Let  $p = m - n$ . Then  $|x_m - x_n| = |x_{n+p} - x_n| \leq |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \dots + |x_{n+1} - x_n| < r^{n+p-1} + r^{n+p-2} + \dots + r^n = r^n \sum_{k=0}^{p-1} r^k = r^n \frac{1-r^p}{1-r} < \frac{r^n}{1-r}$ .  $r^n \rightarrow 0$  as  $n \rightarrow \infty$ , so if  $\epsilon > 0$ , choose  $K$  such that  $r^K < \epsilon(1-r)$ . Then if  $m \geq n \geq K$ , we get  $|x_m - x_n| < \frac{r^K}{1-r} < \epsilon$ , and hence the sequence is Cauchy.

#10) Let  $l = x_2 - x_1$ . First we prove (we will need it later) that  $x_{2k+2} > x_{2k+1}$  for all  $k \geq 0$ . It is true that  $x_2 > x_1$ , so the base case holds. Assume it is true for some  $j$ . We must prove that it is true for  $j + 1$ ,

that is: prove  $x_{2j+4} > x_{2j+3}$ . To show this, we note that  $x_{2j+4} = \frac{1}{2}(x_{2j+3} + x_{2j+2}) = \frac{1}{4}(x_{2j+1} + 3x_{2j+2})$  and  $x_{2j+3} = \frac{1}{2}(x_{2j+1} + x_{2j+2})$ . So  $x_{2j+4} - x_{2j+3} = \frac{1}{4}(x_{2j+2} - x_{2j+1}) > 0$  by assumption. Hence, our claim is proven by induction.

Now, for any  $n > 1$ ,  $|x_{n+1} - x_n| = \frac{1}{2}|x_n - x_{n-1}|$ , and so the sequence is contractive. Moreover, we can expand this result by induction to get  $|x_{n+1} - x_n| = \frac{1}{2^{n-1}}|x_2 - x_1| = \frac{l}{2^{n-1}}$ . Since the sequence is contractive, we know that it is Cauchy, whence convergent. Let its limit be  $x$ . We examine the odd-indexed subsequence  $(x_{2k+1})$ , and we claim that  $x_{2k+1} = x_1 + \sum_{j=0}^k \frac{l}{2^{2j-1}}$ . For  $k = 1$ , this is true since  $x_3 = \frac{1}{2}(x_2 + x_1) = \frac{1}{2}(l + x_1 + x_1) = x_1 + \frac{l}{2}$  as desired. Assume it is true for some  $k$ ; we will prove it for  $k+1$  and the result follows by induction:  $x_{2(k+1)+1} = x_{2k+3} = \frac{1}{2}(x_{2k+2} + x_{2k+1})$ . We know that  $|x_{2k+2} - x_{2k+1}| = \frac{l}{2^{2k}}$ . Moreover, we know that  $x_{2k+2} > x_{2k+1}$  and so  $x_{2k+2} = x_{2k+1} + \frac{l}{2^{2k}}$ . It follows that  $x_{2k+3} = \frac{1}{2}(2x_{2k+1} + \frac{l}{2^{2k}}) = x_{2k+1} + \frac{l}{2^{2k+1}} = x_1 + \sum_{j=0}^k \frac{l}{2^{2j-1}} + \frac{l}{2^{2k+1}} = x_1 + \sum_{j=0}^{k+1} \frac{l}{2^{2j-1}}$  as desired. Therefore,

$$x_{2k+1} = x_1 + \sum_{j=0}^k \frac{l}{2^{2j-1}} = x_1 + \frac{l}{2} \sum_{j=0}^{k-1} \frac{1}{4^j} = x_1 + \frac{l}{2} \left( \frac{1 - (\frac{1}{4})^k}{1 - \frac{1}{4}} \right) = x_1 + \frac{2l}{3} (1 - (1/4)^k)$$

Taking the limit of both sides yields  $x = x_1 + \frac{2l}{3} = \frac{1}{3}x_1 + \frac{2}{3}x_2$ .

**#12)** Note that  $x_n > 0$  for all  $n$  so that  $5 + 2x_n > 5$ .

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left| \frac{1}{2 + \frac{1}{2+x_n}} - \frac{1}{2+x_n} \right| = \left| \frac{2+x_n}{5+2x_n} - \frac{1}{2+x_n} \right| \\ &= \left| \frac{4+4x_n+x_n^2-5-2x_n}{(5+2x_n)(2+x_n)} \right| \leq \frac{1}{5} \left| \frac{-1+2x_n+x_n^2}{2+x_n} \right| = \frac{1}{5} \left| \frac{-1}{2+x_n} + x_n \right| \\ &= \frac{1}{5} |x_n - x_{n+1}| = \frac{1}{5} |x_{n+1} - x_n| \end{aligned}$$

Since  $\frac{1}{5} < 1$ , this shows the sequence is contractive. Its limit must be positive and satisfy  $x = \frac{1}{2+x}$ , and so the limit is  $x = \sqrt{2} - 1$ .