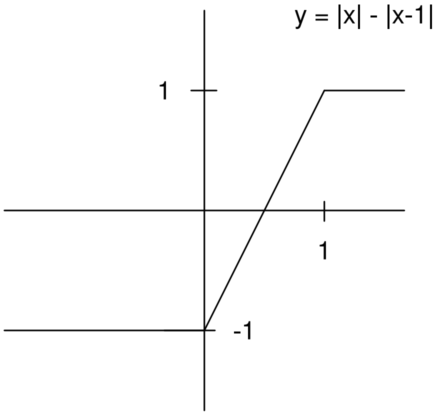
Section 2.2

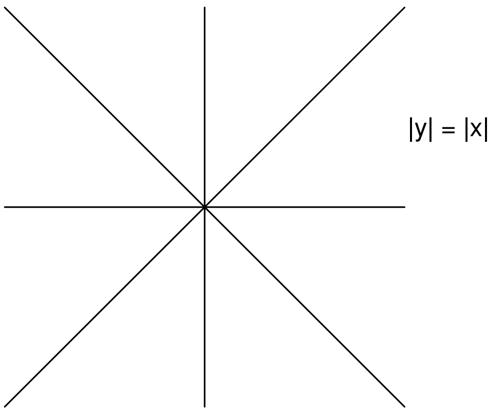
#3) If $x \leq y \leq z$, then $|x-y|+|y-z| = y-x+z-y = z-x = |x-z|$. If $y < x \leq z$, then $|y-z| = z-y > z-x = |x-z|$; therefore $|x-y|+|y-z| > |z-x|$ since $|x-y| > 0$. Finally, if $x \leq z < y$, then $|y-x| = y-x > z-x = |x-z|$; therefore $|x-y|+|y-z| > |z-x|$ since $|x-y| > 0$. Geometrically, the triangle inequality is actually equality if and only if y is on the line between x and z.

#7) If $x \geq 2$, then $|x+1|+|x-2| = x+1+x-2 = 2x-1$. $2x-1=7$ has solution $x=4$ which indeed lies in the region $x \geq 2$. If $-1 \leq x < 2$, then $|x+1|+|x-2| = x+1+2-x = 3$. $3 \neq 7$, so there is no solution in this range. Finally, if $x < -1$, then $|x+1|+|x-2| = -x-1+2-x = 1-2x$. $1-2x=7$ has solution $x=-3 < -1$. Thus, the equation has two solutions: $x=4$ and $x=-3$.

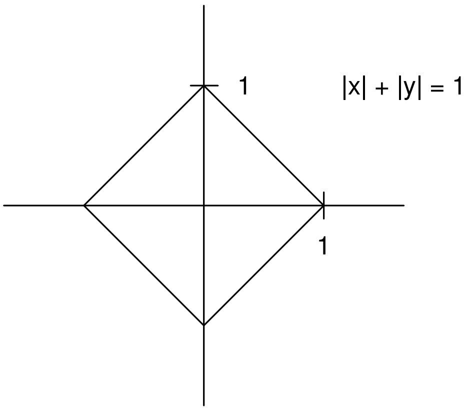


#9)

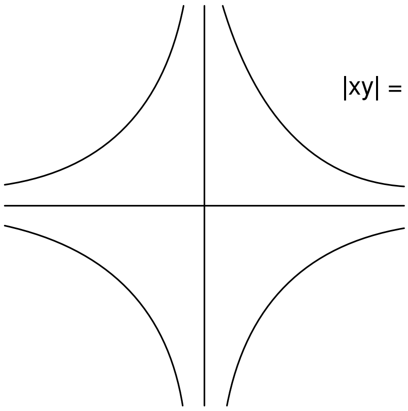
#12) (a) $\{(t,t)$ and $(t,-t)$: $t \in \mathbb{R}\}$.



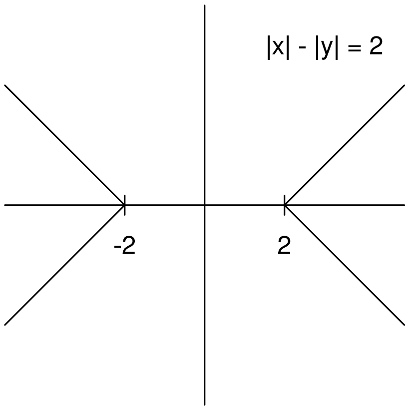
(b) $\{(t, 1-t),\ (t, t-1),\ (-t, 1-t),$ and $(-t, t-1)\colon\ 0 \le t \le 1\}$.



(c) $\{(t, 2/t)$ and $(t, -2/t)\colon\ t \ne 0\}$.



(d) $\{(t, 2-t),\ (t, t-2),\ (-t, 2-t),$ and $(-t, t-2)\colon\ t \ge 2\}$.



#14) Let $\gamma_1 = \min(\epsilon, \delta)$ and let $\gamma_2 = \max(\epsilon, \delta)$. We claim $V_{\gamma_1}(a) = V_\epsilon(a) \cap V_\delta(a)$ and $V_{\gamma_2}(a) = V_\epsilon(a) \cup V_\delta(a)$.

$x \in V_{\gamma_1}(a) \iff |x-a| < \gamma_1 = \min(\epsilon, \delta) \iff |x-a| < \epsilon$ and $|x-a| < \delta \iff x \in V_\epsilon(a)$ and $x \in V_\delta(a) \iff x \in V_\epsilon(a) \cap V_\delta(a)$.

$x \in V_{\gamma_2}(a) \iff |x-a| < \gamma_2 = \max(\epsilon, \delta) \iff |x-a| < \epsilon$ or $|x-a| < \delta \iff x \in V_\epsilon(a)$ or $x \in V_\delta(a) \iff x \in V_\epsilon(a) \cup V_\delta(a)$.

#15) Let $\epsilon = |a-b|/2$. Suppose $x \in V_\epsilon(a) \cap V_\delta(a)$. Then $|x-a| < \epsilon$ and $|x-b| < \epsilon$. Therefore, by the triangle inequality, $|a-b| = |a-x+x-b| \leq |a-x| + |x-b| < 2\epsilon = |a-b|$. Contradiction. Thus, $V_\epsilon(a) \cap V_\delta(a) = \emptyset$.

Section 2.3

#4) $\sup S_4 = 2$. First, $2 \in S_4$, so if 2 is an upper bound, it will automatically be the least upper bound since it will be at least as small as every other upper bound. Now, if $n \in \mathbb{N}$ is even, then $1-(-1)^n/n = 1-1/n < 1 < 2$; if n is odd, then $1-(-1)^n/n = 1+1/n \leq 2$ since $1/n \leq 1$. Thus, 2 is indeed an upper bound.

$\inf S_4 = 1/2$. First, $1/2 \in S_4$, so if it is a lower bound, it will automatically be the greatest lower bound since it will be at least as large as every other lower bound. Now if $n \in \mathbb{N}$ is odd, then $1-(-1)^n/n = 1+1/n > 1 > 1/2$; if n is even, then $1/n \leq 1/2$, so $1-(-1)^n/n = 1-1/n \geq 1/2$. Thus, 1/2 is indeed a lower bound.

#5) Suppose S is bounded below. Let l be a lower bound of S. Then $l \leq s$ for all $s \in S$. So $-l \geq -s$ for all $s \in S$. So $-l$ is an upper bound for $\{-s : s \in S\}$. So $u = \sup\{-s : s \in S\}$ exists. Since u is an upper bound for this set, a similar argument as the one given for l shows that $-u$ is a lower bound for $\{-(-s) : s \in S\} = S$. Also, l' is a lower bound for $S \iff -l'$ is an upper bound for $\{-s : s \in S\}$ $\iff -l \geq u \iff l \leq -u$, showing that $-u = \inf S$ as desired.

#8) Let $n \in \mathbb{N}$. $1/n > 0$, so $u-1/n < u$. Therefore, by 2.3.3 there exists $s \in S$ such that $u-1/n < s$. Thus, $u-1/n$ is not an upper bound. $u+1/n > u$. u is an upper bound, so for all $s \in S$, $s \leq u < u+1/n$. Thus, $u+1/n$ is an upper bound.

#11) First, $u \neq s^*$ since $u \notin S$ and $s^* \in S$. If $u > s^*$, then $u = \sup\{s^*, u\}$ since u is an upper bound of this set and also contained in the set. Likewise, if $s^* > u$, then $s^* = \sup\{s^*, u\}$. If $u > s^*$, then $u > s^* \geq s$ for all $s \in S$; so u is an upper bound for $S \cup \{u\}$. Since u is also contained in this set, it must be the least upper bound. Likewise, if $s^* > u$, then s^* is an upper bound of $S \cup \{u\}$ since it was already an upper bound of S. Again, since $s^* \in S$, s^* must be the least upper bound in this case.

#12) By induction. Base Case: if $|S| = 1$, then $S = \{a\}$ for some $a \in \mathbb{R}$. Clearly, a is an upper bound of S, since $a \geq a$. Furthermore, if u is an upper bound of S, then $u \geq a$, proving that $a = \sup S$.

Now assume that for some $k \in \mathbb{N}$, if $|S| = k$, then $\sup S \in S$. Let $|T| = k+1$, and let $x \in T$, $S = T \backslash \{x\}$. Then $x \notin S$, $|S| = k$. So $\sup S \in S \subseteq T$. By Problem 11, $\sup T = \sup S \cup \{x\} = \sup\{\sup S, x\}$, which is either $\sup S$ or x. Therefore, $\sup T \in T$ as required.