Section 5.6

#1) First suppose f is increasing. Then for any $x \in [a, b]$, $a \leq x \leq b$, and hence $f(a) \leq f(x) \leq f(b)$. Hence, $f(a)$ is an absolute minimum and $f(b)$ is an absolute maximum.

If f is strictly increasing, then $x \in (a, b) \implies a < x < b \implies f(a) < f(x) < f(b)$, and so $f(a)$ and $f(b)$ are the unique absolute minimum and maximum.

#5) First suppose that f is continuous at a. By Problem 1), $f(a)$ is a lower bound for $\{f(x) : x \in (a, b]\}$. We must show it is the greatest lower bound, or equivalently, show that for all $\epsilon > 0$, there exists $x_0 \in (a, b]$ such that $f(x_0) < f(a) + \epsilon$. Fix $\epsilon > 0$. Since f is continuous at a, we can find $\delta > 0$ such that if $x \in [a, b]$ and $|x - a| < \delta$, then $f(a) - \epsilon < f(x) < f(a) + \epsilon$. Therefore, choosing any $a < x_0 < a + \delta$ will work.

Conversely, suppose that $f(a) = \inf\{f(x) : x \in (a, b]\}$. Fix $\epsilon > 0$. We can find $x_0 \in (a, b]$ such that $f(a) \leq f(x_0) < f(a) + \epsilon$. Set $\delta = x_0 - a > 0$. Suppose $x \in [a, b]$, and $|x - a| < \delta$. Then $a \leq x < a + \delta = x_0$. Therefore, since f is increasing, $f(a) \leq f(x) \leq f(x_0) < f(a) + \epsilon$, by the definition of x_0. Ergo, $|f(x) - f(a)| < \epsilon$, and f is continuous at a.

#8) Suppose $f(x_0) = y = g(x_1)$ for some $x_0, x_1 \in I$. If $x_0 \geq x_1$, then $y = f(x_0) \geq f(x_1)$ since f is increasing. But $f(x_1) > g(x_1) = y$, so we get $y > y$, contradiction. Hences, $x_0 < x_1$.

#9) If x is rational, then $f(x) = x$ is rational. If x is irrational, then $f(x) = 1 - x$ is irrational. Suppose therefore, that $f(x_1) = f(x_2) = y$. If y is rational, then x_1, x_2 must both be rational and hence $y = x_1 = x_2$. If y is irrational, then x_1, x_2 must both be irrational, and hence $y = 1 - x_1 = 1 - x_2$, which automatically implies that $x_1 = x_2$. Therefore, f is injective.

Note that if $0 \leq x \leq 1$, then $0 \leq 1 - x \leq 1$ as well. So the composition $f \circ f$ is well-defined. If x is rational, then $f(f(x)) = f(x) = x$. If x is irrational, then $f(f(x)) = f(1 - x) = 1 - (1 - x) = x$. So $f \circ f$ is the identity.

Let $\epsilon > 0$, and set $\delta = \epsilon$. Suppose $x \in [0, 1]$ and $|x - 1/2| < \delta$. If x is rational, then $|f(x) - f(1/2)| = |x - 1/2| < \epsilon$. If x is irrational, then still, $|f(x) - f(1/2)| = |1 - x - 1/2| = |1/2 - x| < \epsilon$. Hence, f is continuous at $1/2$.

Let $c \neq 1/2$. Let (x_n) be a sequence of rationals converging to c. Let (y_n) be a sequence of irrationals converging to c. Then $f(x_n) = x_n \to c$ and $f(y_n) = 1 - y_n \to 1 - c$. Since $c \neq 1/2$, $c \neq 1 - c$. Therefore, the limit of f at c does not even exist. f is not continuous at c.

#13) Suppose for contradiction that h is continuous. Then by the Max-Min principle, there is are points where h achieves an absolute maximum M and an absolute minimum m. By assumption, h achieves its absolute maximum in two places $c_1 < c_2$. We claim that $c_1 = 0$. Suppose not. Then $h(0) < M$. Fix z such that $c_1 < z < c_2$. Then $h(z) < M$ as well. Choose $k < M$ such that $k > f(0)$ and $k > f(z)$. Then by the Intermediate Value Theorem, there exist $0 < x_1 < c_1 < x_2 < z < x_{<}c_2$ such that $h(x_1) = h(x_2) = h(x_3) = k$. This contradicts the assumption that h takes the value k exactly twice. Hence $c_1 = 0$ and so $h(0) = M$.

The above paragraph shows that if f is a continuous function on $[0, 1]$ that takes each of its values twice, then it must take its maximum at 0. h is such a function, but so is $-h$. Hence $-h$ takes is maximum at 0, but the maximum of $-h$ is $-m$. Therefore, $-h(0) = -m$, and hence $h(0) = m$. Therefore, $m = M$; that is, the minimum value of h is the same as the maximum value. Therefore, h is a constant function: $h(x) = M$ for all x. But then h takes the value M an infinite number of times, contrary to the hypothesis.

#15) If a and b are integers, then we already know that $x^a x^b = x^{a+b}$.

Let m, n, p, q be integers such that $r = m/n$ and $s = p/q$. By problem 14, $x^r = x^{m/n} = x^{mq/nq}$. Similarly, $x^s = x^{np/nq}$. Therefore,

$$x^r x^s = x^{mq/nq} x^{np/nq} = (x^{1/nq})^{nq} (x^{1/nq})^{np} = (x^{1/nq})^{mq+np} = x^{(mq+np)/nq} = x^{r+s}.$$

The second and fourth equality are the definition of rational exponents, and the third is the property of integer exponents. The second equality comes from switching the role of r and s and noting that addition is commutative.

To prove that $(x^r)^s = x^{rs}$, first note that we have this property for integer exponents. Then it follows that $((x^{1/n})^{1/q})^{nq} = (((x^{1/n})^{1/q})^q)^n = (x^{1/n})^n = x$. So $(x^{1/n})^{1/q} = x^{1/(nq)}$. Now we can compute using Theorem 5.6.7:

$$(x^r)^s = (x^{m/n})^{p/q} = (((x^{1/n})^m)^{1/q})^p = (((x^{1/n})^{1/q})^m)^p = (x^{1/nq})^{mp} = x^{mp/nq} = x^{rs}.$$

Section 6.1

#1) (a) Note that for $x \neq c$, $\frac{x^3-c^3}{x-c} = x^2 + xc + c^2$. Therefore, $f'(c) = \lim_{x\to c} \frac{x^3-c^3}{x-c} = 3c^2$.

(b) Note that for $x \neq c$ and $x, c \neq 0$, $\frac{1/x-1/c}{x-c} = \frac{-1}{xc}$. Therefore, $f'(c) = \lim_{x\to c} \frac{1/x-1/c}{x-c} = -1/c^2$.

(c) Note that for $x \neq c$ and $x, c > 0$, $\frac{\sqrt{x}-\sqrt{c}}{x-c} = 1/(\sqrt{x}+\sqrt{c})$. Therefore, $f'(c) = \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c} = 1/(2\sqrt{c})$.

(d) Note that for $x \neq c$ and $x, c > 0$, $\frac{1/\sqrt{x}-1/\sqrt{c}}{x-c} = frac{-1}\sqrt{xc}(\sqrt{x}+\sqrt{c})$. Therefore, $f'(c) = \lim_{x\to c} \frac{1/\sqrt{x}-1/\sqrt{c}}{x-c} = \frac{-1}{2c^{3/2}}$.

#2) We need to show that $\lim \frac{x^{1/3}-0^{1/3}}{x-0}$ does not exist. It suffices to show that $x^{-2/3}$ is not bounded, but this is obvious.

#7) Suppose $f'(c) = 0$. Then $\lim_{x\to c} \frac{f(x)}{x-c} = 0$. Then $\lim_{x\to c} \left|\frac{f(x)}{x-c}\right| = 0$. Then $\lim_{x\to c} \left|\frac{|f(x)|}{x-c}\right| = 0$. $\lim_{x\to c} \left|\frac{g(x)}{x-c}\right| = 0$. Then $\lim_{x\to c} \frac{g(x)}{x-c} = 0$. Therefore, $g'(c)$ exists, and equals 0.

Inversely, suppose $f'(c) = L \neq 0$. First, we deal with the case when $L > 0$. So $\lim_{x\to c} \frac{f(x)}{x-c} = L > 0$. Therefore, we can find $\delta > 0$ such that if $0 < |x-c| < \delta$, then $\frac{f(x)}{x-c} > 0$; i.e. $f(x)$ and $x - c$ have the same sign for $0 < |x-c| < \delta$. Therefore, for $c < x < c+\delta$, $f(x)$ is positive and $g(x) = f(x)$. For $c - \delta < x < c$, $f(x)$ is negative and $g(x) = -f(x)$. Thus, $\lim_{x\to c^+} \frac{g(x)}{x-c} = \lim_{x\to c^+} \frac{f(x)}{x-c} = L$, but $\lim_{x\to c^-} \frac{g(x)}{x-c} = \lim_{x\to c^-} \frac{-f(x)}{x-c} = -L$. Since $L \neq -L$, g is not differentiable at c. If $L < 0$, then we apply the above argument to $-f$ since $-f'(c) = -L > 0$ and $g(x) = |f(x)| = |-f(x)|$.

#9) $f'(-c) = \lim_{x\to -c} \frac{f(x)-f(-c)}{x+c} = \lim_{x\to c} \frac{f(-x)-f(-c)}{-x+c} = \lim_{x\to c} \frac{f(x)-f(c)}{-(x-c)} = -f'(c)$. Similarly, $g'(-c) = \lim_{x\to -c} \frac{g(x)-g(-c)}{x+c} = \lim_{x\to c} \frac{g(-x)-g(-c)}{-x+c} = \lim_{x\to c} \frac{-g(x)+g(c)}{-x+c} = g'(c)$.

#11) (a) $f'(x) = L'(2x+3)(2x+3)' = \frac{2}{2x+3}$

(b) $g'(x) = 3(L(x^2))^2 L'(x^2)(2x) = 6(L(x^2))^2/x$

(c) $h'(x) = L'(ax)a = a/ax = 1/x$

(d) $k'(x) = L'(L(x))L'(x) = 1/(xL(x))$