EXAMPLE OF USE OF THE DEFINITION OF LIMIT OF A SEQUENCE

1. Show that

$$\lim\left(\frac{2n}{n+2}\right) = 2.$$

Solution.

$$\left|\frac{2n}{n+2} - 2\right| = \left|\frac{4}{n+2}\right| \leq \left|\frac{4}{n}\right| \qquad (*).$$

Fix $\epsilon > 0$.
By the Archimedean property, there is N_ϵ such that

$$\frac{4}{\epsilon} < N_\epsilon.$$

We thus have

$$\frac{4}{n} < \epsilon \qquad \forall n \geq K := N_\epsilon.$$

By plugging this into $(*)$, we see that, for every $\epsilon > 0$, there is a natural number K such that if $n \geq K$, then

$$\left|\frac{2n}{n+2} - 2\right| < \epsilon$$

and this means that $\lim\left(\frac{2n}{n+2}\right) = 2$.

2. Prove that if $(a_n) \to A$ and $(b_n) \to B$, then

$$(a_n + b_n) \longrightarrow (A + B).$$

Solution. By regrouping the terms and by the the Triangle Inequality:

$$|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| \qquad (*).$$

Fix $\epsilon > 0$. By assumption there are $K_1, K_2 \in \mathbb{N}$ such that

$$|a_n - A| < \frac{\epsilon}{2},\ \forall n \geq K_1, \qquad |b_n - B| < \frac{\epsilon}{2},\ \forall n \geq K_2.$$

Take $K := \max\{K_1, K_2\}$. We have

$$|a_n - A| + |b_n - B| < \epsilon,\ \forall n \geq K \qquad (**).$$

By combining $(*)$ with $(**)$, we obtain that, for any $\epsilon > 0$, there is $K \in \mathbb{N}$ such that

$$|(a_n + b_n) - (A + B)| < \epsilon,\ \forall n \geq K,$$

that is, we have showed that $(a_n + b_n) \to (A + B)$.