SAMPLE MIDTERM II

1 Use axioms V p 374-5 to show that

a) the additive inverse of any $x \in \mathbb{R}$ is unique.

b) $-(-x) = x$.

2 Use induction to prove that $(1^3 + 2^3 + \dots + n^3) = (1 + 2 + \dots + n)^2$.

3 Re-write the set $\{x \in \mathbb{R} \mid |x^2+1| < 2\}$ in interval notation.

4 Consider the statement $\forall x, y \; (x^2 - x \geq -|y|)$.

a) Is it true when the domain is $\mathbb{R}$?

b) " " $\mathbb{Z}$?

5) Show that $\mathbb{N} \cap \mathbb{Q} = \mathbb{Z} \cap \mathbb{N}$

6) Set $R := \{(x,y) \mid x - y^2 = 1\}$; $S = \{(x,y) \mid x^2 + y^2 = 0\}$

Find

i) Dom(R)

ii) Dom(S)

iii) Rng(R)

iv) Rng(S)

v) R^{-1}

Solutions

1.a Let y, z be 2 inverses to x: $x+y=0$, $x+z=0$.

So $x+y = x+z$.

Add y to both $x+y+y = x+z+y$.

Since $x+y=0$ $y = x+z+y$.

By commutativity/assoc $y = x+y+z$

Since $x+y=0$ $y = z$.

1.b $x+(-x)=0$

By commutativity $(-x)+x=0$.

By uniqueness 1.a $x = -(-x)$.

2 $n=1$: $(1^3) = (1)^2$ ok.

Assume $(1^3+\dots+n^3) = (1+\dots+n)^2$

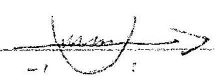
$$1^3+\dots+n^3+(n+1)^3 = (1^3+\dots+n^3)+(n+1)^3 = (1+\dots+n)^2+(n+1)^3.$$

$$(1+\dots+n+(n+1))^2 = (1+\dots+n)^2 + 2(1+\dots+n)(n+1) + (n+1)^2$$

$$= (1+\dots+n)^2 + \cancel{2}\tfrac{1}{\cancel{2}}\, n(n+1)^2 + (n+1)^2 = (1+\dots+n)^2 + (n+1)^2(n+1)$$

$$= (1+\dots+n)^2 + (n+1)^3.$$

3 $x^2+1 \geq 0$ so $|x^2+1| = x^2+1$ so $|x^2+1| < 2$ is $x^2+1<2$ is $x^2-1<0$.



Answer: $(-1,1)$.

4 $x^2 - x = x(x-1)$

a) FALSE: $x=\frac{1}{2}$ $y=0$ then $\frac{1}{2}(\frac{1}{2}-1) \geq -|0|$ is $-\frac{1}{4} \geq 0$ false.

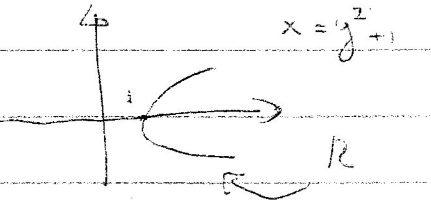
b) TRUE: $\forall x \in \mathbb{Z}$ $x(x-1) \geq 0$, $-|y| \leq 0$.

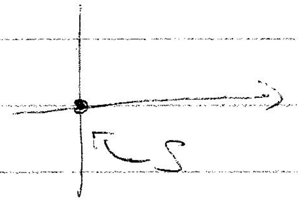
5 We show that i) $\mathbb{N} \cap \mathbb{Q} \supseteq \mathbb{Z} \cap \mathbb{N}$ & ii) $\mathbb{N} \cap \mathbb{Q} \subseteq \mathbb{Z} \cap \mathbb{N}$.

i) $\mathbb{N} \supseteq \mathbb{N}$ & $\mathbb{Q} \supseteq \mathbb{Z}$ so $\mathbb{N} \cap \mathbb{Q} \supseteq \mathbb{Z} \cap \mathbb{N}$

ii) if $x \in \mathbb{N}$ & $x \in \mathbb{Q}$ then $x \in \mathbb{N}$ & $x \in \mathbb{Z}$ so $\mathbb{N} \cap \mathbb{Q} \subseteq \mathbb{Z} \cap \mathbb{N}$. $\square$

6





i) $[1, \infty)$

ii) $\{0\}$

iii) $(-\infty, +\infty)$

iv) $\{0\}$

v) $R^{-1} = \{ (x, y) \mid y - x^2 = 1 \}$